

# P-ADIC PERIODS AND DERIVED DE RHAM COHOMOLOGY

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*To Irene*

## INTRODUCTION.

For a smooth variety  $X$  over a base field of characteristic 0 we have its algebraic de Rham cohomology  $H_{\mathrm{dR}}^*(X) := H^*(X_{\mathrm{Zar}}, \Omega_X^*)$ ; for nonsmooth  $X$ , one defines  $H_{\mathrm{dR}}^*(X)$  using cohomological descent as in Deligne [D]. If the base field is  $\mathbb{C}$ , then one has the Betti cohomology  $H_{\mathrm{B}}^*(X) := H^*(X_{\mathrm{cl}}, \mathbb{Q})$  and a canonical period isomorphism (“integration of algebraic differential forms over topological cycles”)

$$(0.1) \quad \rho : H_{\mathrm{dR}}^*(X) \xrightarrow{\sim} H_{\mathrm{B}}^*(X) \otimes \mathbb{C}$$

compatible with the  $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ -conjugation. To define  $\rho$ , consider the analytic de Rham cohomology  $H_{\mathrm{dR}}^*(X_{\mathrm{an}})$ . There are evident maps

$$(0.2) \quad H_{\mathrm{dR}}^*(X) \xrightarrow{\alpha} H_{\mathrm{dR}}^*(X_{\mathrm{an}}) \xleftarrow{\beta} H_{\mathrm{B}}^*(X) \otimes \mathbb{C}.$$

Then  $\beta$  is an isomorphism due to the Poincaré lemma, and  $\rho := \beta^{-1}\alpha$  (the fact that  $\rho$  is an isomorphism was established by Grothendieck [Gr]).

Suppose our base field is an algebraic closure  $\bar{K}$  of a  $p$ -adic field  $K$  (say,  $K = \mathbb{Q}_p$ ). The role of  $H_{\mathrm{B}}^*(X)$  is now played by the  $p$ -adic étale cohomology  $H_{\mathrm{ét}}^*(X, \mathbb{Q}_p)$ , and Fontaine conjectured in [F1]<sup>1</sup> the existence of a natural  $p$ -adic period isomorphism

$$(0.3) \quad \rho : H_{\mathrm{dR}}^*(X) \otimes_{\bar{K}} B_{\mathrm{dR}} \xrightarrow{\sim} H_{\mathrm{ét}}^*(X, \mathbb{Q}_p) \otimes B_{\mathrm{dR}}.$$

Here  $B_{\mathrm{dR}}$  is Fontaine’s  $p$ -adic periods field ([F1], [F3]). Recall that it is a complete discretely-valued field whose ring of integers  $B_{\mathrm{dR}}^+$  contains  $\bar{K}$ , the residue field  $B_{\mathrm{dR}}^+/\mathfrak{m}_{\mathrm{dR}}$  is Tate’s field  $\mathbb{C}_p$ , the cotangent line  $\mathfrak{m}_{\mathrm{dR}}/\mathfrak{m}_{\mathrm{dR}}^2$  is the Tate twist  $\mathbb{C}_p(1)$ . Both sides of (0.3) carry natural filtrations (coming from the filtration of  $B_{\mathrm{dR}}$  by powers of  $\mathfrak{m}_{\mathrm{dR}}$  and the Hodge-Deligne filtration on  $H_{\mathrm{dR}}^*(X)$ ), and  $\rho$  is compatible with them and with the  $\mathrm{Gal}(\bar{K}/K)$ -conjugation. Moreover, as was envisioned by Fontaine and Jannsen ([F4], [J]), the matrix coefficients of  $\rho$  lie in the subring  $\bar{K}B_{\mathrm{st}}$  of  $B_{\mathrm{dR}}$ ,<sup>2</sup> and  $\rho$  is compatible with the extra symmetries of log crystalline cohomology.

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<sup>1</sup>The assumption of loc. cit. that  $X$  is proper and smooth is redundant.

<sup>2</sup>This assertion is true for a general reason, since, by Berger’s theorem [Ber], de Rham Galois modules are potentially log crystalline.

The p-adic period map was defined in three different ways in works of, respectively, Faltings, Niziol, and Tsuji (with prior crucial input of Bloch, Fontaine, Hyodo, Kato, Kurihara, and Messing; the nonproper setting was treated by Yamashita [Y]), see [Fa1], [Fa2], [N1], [N2], [Ts1], [Ts2]; the three  $\rho$ 's coincide by [N3].

In the article we give another construction of  $\rho$  which is fairly direct and has the same flavor as the classical picture (0.2). The tools are derived de Rham cohomology of Illusie [Ill2] Ch. VIII and de Jong's alterations. The companion paper [B] treats the Fontaine-Jannsen side of the story; another approach was developed by Bhatt [Bh2]. It would be very interesting to see if these methods can help to understand the Riemann-Hilbert correspondence in the p-adic setting.

An outline of the construction: First we realize  $B_{\text{dR}}^+$  as the ring of de Rham p-adic constants in the sense of *derived* algebraic geometry. Namely, let  $A_{\text{dR}}$  be the derived de Rham algebra  $L\Omega_{\hat{O}_{\bar{K}}/O_K}$  completed with respect to the Hodge filtration  $F^\cdot$ , see [Ill2] Ch. VIII (2.1.3.3); here  $O_K, O_{\bar{K}}$  are the rings of integers in  $K, \bar{K}$ . Now  $B_{\text{dR}}^+$  identifies canonically with  $A_{\text{dR}} \hat{\otimes} \mathbb{Q}_p$ , where  $\hat{\otimes}$  is the derived completed tensor product, so that  $\mathfrak{m}_{\text{dR}}^i \xrightarrow{\sim} F^i A_{\text{dR}} \hat{\otimes} \mathbb{Q}_p$ . This fact was observed independently by Fargues [Far].

Let  $\mathcal{V}ar_F$  be the category of varieties over a field  $F$ , and  $\mathcal{V}ar_F^{\text{nc}}$  be the category of regular  $F$ -varieties  $U$  equipped with a regular compactification  $\bar{U}$  with normal crossings divisor at infinity. As follows from de Jong's theorem [dJ1], the forgetful functor  $\mathcal{V}ar_F^{\text{nc}} \rightarrow \mathcal{V}ar_F, (U, \bar{U}) \mapsto U$ , makes  $\mathcal{V}ar_F^{\text{nc}}$  a base for the h-topology on  $\mathcal{V}ar_F$ , so h-sheaves on  $\mathcal{V}ar_F$  are the same as sheaves on  $\mathcal{V}ar_F^{\text{nc}}$  for the induced topology. For  $F = \bar{K}$  as above, there is a finer category  $\mathcal{V}ar_{\bar{K}}^{\text{ss}}$  of *ss-pairs*  $(V, \bar{V})$ , i.e., smooth  $\bar{K}$ -varieties  $V$  equipped with a semi-stable compactification  $\bar{V}$  (that includes compactification in the arithmetic direction). Again by de Jong [dJ1],  $\mathcal{V}ar_{\bar{K}}^{\text{ss}}$  is a base for the h-topology on  $\mathcal{V}ar_{\bar{K}}$ .

Consider the presheaf on  $\mathcal{V}ar_{\bar{K}}^{\text{ss}}$  which assigns to  $(V, \bar{V})$  the derived de Rham algebra with log singularities  $R\Gamma(\bar{V}, L\Omega_{(\bar{V}, \bar{V})/O_K}^\wedge)$  (see [Ol]). Its h-sheafification  $\mathcal{A}_{\text{dR}}^\natural$  is an h-sheaf of filtered dg algebras on  $\mathcal{V}ar_{\bar{K}}$  that contains the constant subsheaf  $A_{\text{dR}}$ . The key *p-adic Poincaré lemma* says that *the map  $A_{\text{dR}} \otimes^L \mathbb{Z}/p^n \rightarrow \mathcal{A}_{\text{dR}}^\natural \otimes^L \mathbb{Z}/p^n$  is a filtered quasi-isomorphism*. It comes from the next assertion: The h-sheafification of the presheaf  $(V, \bar{V}) \mapsto H^b(\bar{V}, \Omega_{(V, \bar{V})/O_K}^a)$ , where  $\Omega_{(V, \bar{V})/O_K}^a$  is the usual locally free  $\mathcal{O}_{\bar{V}}$ -module of forms with log singularities, is an h-sheaf of  $\mathbb{Q}$ -vector spaces for  $(a, b) \neq (0, 0)$ . The case  $a = 0$  is essentially theorem 8.0.1 from Bhatt's thesis [Bh1]; the general result is obtained by a similar method (which uses coverings of families of stable curves that come from the multiplication by  $p$  isogeny of the generalized Jacobians).

Set  $R\Gamma_{\text{dR}}^\natural(X) := R\Gamma(X_{\text{h}}, \mathcal{A}_{\text{dR}}^\natural)$ ; this is the *arithmetic de Rham complex* of  $X$ . By the above,  $H^\cdot(R\Gamma_{\text{dR}}^\natural(X) \hat{\otimes} \mathbb{Q}_p)$  is a  $B_{\text{dR}}^+$ -algebra. One has a diagram

$$(0.4) \quad H_{\text{dR}}(X) \xrightarrow{\alpha} H^\cdot(R\Gamma_{\text{dR}}^\natural(X) \hat{\otimes} \mathbb{Q}_p) \xleftarrow{\beta} H_{\text{ét}}(X, \mathbb{Q}_p) \otimes B_{\text{dR}}^+,$$

where  $\alpha$  is the composition  $H_{\text{dR}}(X) \xrightarrow{\sim} H^\cdot R\Gamma_{\text{dR}}^\natural(X) \otimes \mathbb{Q} \rightarrow H^\cdot(R\Gamma_{\text{dR}}^\natural(X) \hat{\otimes} \mathbb{Q}_p)$  and  $\beta$  is the  $B_{\text{dR}}^+$ -linear extension of the evident map (which comes from the embeddings  $\mathbb{Z}/p^n \rightarrow \mathcal{A}_{\text{dR}}^\natural \otimes^L \mathbb{Z}/p^n$  and the fact that the h-topology is stronger than the étale one). Since the étale and h-cohomology with torsion coefficients coincide, the Poincaré lemma implies that  $\beta$  is an isomorphism. Now the p-adic period map  $\rho$  is

the  $B_{dR}$ -linear extension of  $\beta^{-1}\alpha$ . An explicit computation for  $X = \mathbb{G}_m$  followed by usual tricks of the trade shows that  $\rho$  is a filtered isomorphism.

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## 1. A DERIVED DE RHAM CONSTRUCTION OF $B_{dR}$ .

1.1. *The derived  $p$ -adic completion.* Throughout the article we use (not too heavily)  $E_\infty$  algebras, for which we refer to, say, [HS].<sup>3</sup> Recall that  $E_\infty$  algebras are dg algebras whose product is commutative and associative up to coherent higher homotopies (more formally,  $E_\infty$  algebras are dg algebras for a resolution of the commutative algebra operad). A key fact: for any commutative (more generally,  $E_\infty$ ) cosimplicial dg algebra the corresponding total complex is naturally an  $E_\infty$  algebra. Thus the homotopy limit of a diagram of  $E_\infty$  algebras is an  $E_\infty$  algebra.

For a projective system of complexes of abelian groups  $\dots \xrightarrow{\phi_2} C_2 \xrightarrow{\phi_1} C_1$ , one has  $\text{holim } C_n = \text{Cone}(\text{id} - \phi : \Pi C_n \rightarrow \Pi C_n)[-1]$ , where  $\phi((c_n)) = (\phi_n(c_{n+1}))$ . There is an embedding  $\varprojlim C_n = \text{Ker}(\text{id} - \phi) \hookrightarrow \text{holim } C_n$ . If all  $\phi_n$ 's are surjective, then  $\text{id} - \phi$  is surjective, hence  $\hookrightarrow$  is a quasi-isomorphism. So  $\text{holim}$ , being an exact functor, is the right derived functor of  $\varprojlim$ .

If  $C$  is a projective system of dg algebras, then  $\text{holim } C_n$  is naturally a dg algebra (and the above embedding is an embedding of algebras); if the  $C_n$  are commutative (or, more generally,  $E_\infty$ ) algebras, then  $\text{holim } C_n$  is an  $E_\infty$  algebra.

Let  $p$  be a prime. Consider the projective system of commutative dg algebras  $C_n := \text{Cone}(\mathbb{Z} \xrightarrow{p^n} \mathbb{Z})$ . It is a resolution of the projective system  $\dots \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p$ , so  $\mathbb{Z}_p^\flat := \text{holim } C_n$  is an  $E_\infty$  algebra with  $H^0 \mathbb{Z}_p^\flat = \mathbb{Z}_p$  and acyclic in nonzero degrees. Set  $\mathbb{Q}_p^\flat := \mathbb{Z}_p^\flat \otimes \mathbb{Q}$ . For any complex  $F$  of abelian groups set

$$(1.1.1) \quad F \hat{\otimes} \mathbb{Z}_p := \text{holim}(F \otimes C_n), \quad F \hat{\otimes} \mathbb{Q}_p := (F \hat{\otimes} \mathbb{Z}_p) \otimes \mathbb{Q}.$$

These are dg  $\mathbb{Z}_p^\flat$  and  $\mathbb{Q}_p^\flat$ -modules, so their cohomology groups are  $\mathbb{Z}_p$ - and  $\mathbb{Q}_p$ -modules, and  $F \mapsto F \hat{\otimes} \mathbb{Z}_p, F \hat{\otimes} \mathbb{Q}_p$  are exact functors. If  $F$  is an  $(E_\infty)$  dg algebra, then so are  $F \hat{\otimes} \mathbb{Z}_p$  and  $F \hat{\otimes} \mathbb{Q}_p$ .

*Remark.* One has an evident projective system  $F_{p^n}[1] \rightarrow F \otimes C_n \rightarrow F/p^n F$  of exact triangles; applying  $\text{holim}$ , we get a canonical exact triangle  $\text{holim } F_{p^n}[1] \rightarrow F \hat{\otimes} \mathbb{Z}_p \rightarrow \text{holim}(F/p^n F)$ . Let  $\hat{F} := \varprojlim F/p^n F$  be the  $p$ -adic completion of  $F$  and  $T_p F := \varprojlim F_{p^n}$  be the Tate module of  $F$ . By above, we have a quasi-isomorphism  $\hat{F} \xrightarrow{\sim} \text{holim}(F/p^n F)$ . Thus if  $F$  has no  $p$ -torsion, then  $F \hat{\otimes} \mathbb{Z}_p \xrightarrow{\sim} \hat{F}$ . Similarly, if all components of  $F$  are  $p$ -divisible, then one has quasi-isomorphisms  $T_p F \xrightarrow{\sim} \text{holim } F_{p^n}$  and  $T_p F[1] \xrightarrow{\sim} F \hat{\otimes} \mathbb{Z}_p$ . We see that  $\cdot \hat{\otimes} \mathbb{Z}_p$  is the left derived functor of the  $p$ -adic completion functor and the right derived functor of  $T_p[1]$ .

*Example.* For a scheme  $X$ , its étale  $\mathbb{Z}_p$ - and  $\mathbb{Q}_p$ -cohomology are  $R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) := \text{holim } R\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n) = R\Gamma(X_{\text{ét}}, \mathbb{Z}) \hat{\otimes} \mathbb{Z}_p$ ,  $R\Gamma_{\text{ét}}(X, \mathbb{Q}_p) := R\Gamma(X_{\text{ét}}, \mathbb{Z}) \hat{\otimes} \mathbb{Q}_p$ .<sup>4</sup>

<sup>3</sup>There  $E_\infty$  algebras are called “May algebras”.

<sup>4</sup>Since  $R\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n) = R\Gamma(X_{\text{ét}}, \mathbb{Z}) \otimes^L \mathbb{Z}/p^n$ .

1.2. *The derived de Rham algebra.* For a morphism of commutative rings  $A \rightarrow B$  we denote by  $\Omega_{B/A}$  the relative de Rham complex of  $B$  over  $A$ . This is a commutative dg  $A$ -algebra with components  $\Omega_{B/A}^i = \Lambda_B^i \Omega_{B/A}$ , where  $\Omega_{B/A}$  is the  $B$ -module of relative Kähler differentials; it carries a ring filtration  $F^n = \Omega_{B/A}^{\geq n}$ .

We will use the  $F$ -completed version  $L\Omega_{B/A}^\wedge$  of Illusie's derived de Rham algebra defined in [Ill2] Ch. VIII, (2.1.3.3). To construct it, consider the canonical simplicial resolution  $P = P_A(B)$  of  $B$  from [Ill1] Ch. I, (1.5.5.6). This is a simplicial commutative  $A$ -algebra such that each  $P_i$  is a polynomial  $A$ -algebra. The de Rham complexes  $\Omega_{P_i/A}$  form a simplicial filtered commutative dg  $A$ -algebra, so the corresponding total complex  $L\Omega_{B/A}$  is a filtered commutative dg  $A$ -algebra (see [Ill1] Ch. I, 3.1.3). Now  $L\Omega_{B/A}^\wedge$  is its completion with respect to the filtration  $F^\cdot$ . Here "completion" is understood as mere projective system of quotients modulo  $F^i$ . One has a natural identification  $\mathrm{gr}_F^i L\Omega_{B/A}^\wedge \xrightarrow{\sim} (L\Lambda_B^i(L_{B/A}))[-i]$  compatible with the product; here  $L_{B/A} := \Omega_{P/A} \otimes_P B$  is the relative cotangent complex and  $L\Lambda_B^i$  is the nonabelian left derived functor of the exterior power functor (see Ch. II and I of [Ill1]). For  $A$ -flat  $B$ 's, the construction is compatible with base change. It is compatible with direct limits. If in the above definition we replace  $P$  by any simplicial  $A$ -algebra resolution of  $B$  whose terms are polynomial  $A$ -algebras, then the output is naturally quasi-isomorphic to  $L\Omega_{B/A}^\wedge$ .

The next lemma is a particular case of [Ill1] Ch. I, 4.3.2.1(ii). For a flat  $B$ -module  $T$  we denote by  $B\langle T \rangle$  its divided powers symmetric algebra.

**Lemma.** *The complex  $L\Lambda_B^i(T[1])$  is acyclic off degree  $-i$ . There is a canonical isomorphism of graded  $B$ -algebras compatible with base change*

$$(1.2.1) \quad H^\cdot(L\Lambda_B(T[1])) \xrightarrow{\sim} B\langle T \rangle.$$

1.3. Let  $K$  be a  $p$ -adic field, i.e., a complete discretely-valued field of characteristic zero with perfect residue field  $k$  of characteristic  $p > 0$ ,  $\bar{K}$  be an algebraic closure of  $K$ ,  $O_K, O_{\bar{K}}$  be rings of integers in  $K, \bar{K}$ . Let  $K_0 \subset K$  be the field of fractions of the Witt vectors  $W(k) = O_{K_0}$ , and let  $\mathfrak{a}$  be the fractional ideal in  $\bar{K}$  generated by  $p^{-\frac{1}{p-1}} \mathfrak{D}_{K/K_0}^{-1}$ , where  $\mathfrak{D}_{K/K_0}$  is the different. For an  $O_K$ -algebra  $B$  we often write  $\Omega_B := \Omega_{B/O_K}$ ,  $L\Omega_B^\wedge := L\Omega_{B/O_K}^\wedge$ ,  $L_B = L_{B/O_K}$ , etc.

The next key result is due to Fontaine [F2] Th 1; we include a proof for completeness sake. Consider the map  $\mu_{p^\infty} \subset O_K^\times \xrightarrow{d\log} \Omega_{O_K}$  and its  $O_{\bar{K}}$ -linear extension

$$(1.3.1) \quad (\bar{K}/O_{\bar{K}})(1) = O_{\bar{K}} \otimes \mu_{p^\infty} \rightarrow \Omega_{O_{\bar{K}}}.$$

**Theorem.** *One has  $L_{O_{\bar{K}}} \xrightarrow{\sim} \Omega_{O_{\bar{K}}}$ , and (1.3.1) is surjective with kernel  $(\mathfrak{a}/O_{\bar{K}})(1)$ .*

*Proof.* If  $K'/K$  is a finite extension, then  $O_{K'}/O_K$  is a complete intersection. So, if  $\pi$  is a generator of  $O_{K'}/O_K$ ,  $f(t)$  its minimal polynomial, then  $L_{O_{K'}}$  is the cone of multiplication by  $f'(\pi)$  endomorphism of  $O_{K'}$ ; hence  $L_{O_{K'}} \xrightarrow{\sim} \Omega_{O_{K'}}$ . Passing to the limit, we get the first assertion. Let us prove the second one.

(i) By the above,  $\Omega_{O_{K'}} \simeq O_{K'}/\mathfrak{D}_{K'/K}$ . If  $K''/K'$  is another finite extension, then the standard exact triangle of the cotangent complexes reduces to a short exact sequence  $0 \rightarrow O_{K''} \otimes_{O_{K'}} \Omega_{O_{K'}/O_K} \rightarrow \Omega_{O_{K''}/O_K} \rightarrow \Omega_{O_{K''}/O_{K'}} \rightarrow 0$ .

(ii) Replacing  $K', K$  by  $K, K_0$  and passing to the limit, we get a short exact sequence  $0 \rightarrow O_{\bar{K}} \otimes_{O_K} \Omega_{O_K/O_{K_0}} \rightarrow \Omega_{O_{\bar{K}}/O_{K_0}} \rightarrow \Omega_{O_{\bar{K}}/O_K} \rightarrow 0$ . Thus it suffices to prove the theorem for  $K = K_0$ , which we now assume.

(iii) Set  $T := \text{Ker}((\bar{K}/O_{\bar{K}})(1) \rightarrow \Omega)$ ,  $F := K(\mu_p)$ . The set of  $O_{\bar{K}}$ -submodules of  $(\bar{K}/O_{\bar{K}})(1)$  is totally ordered by inclusion. Thus, since  $O_{\bar{K}} \otimes_{O_F} \Omega_{O_F} \subset \Omega_{O_{\bar{K}}}$  is a nonzero  $O_{\bar{K}}$ -module generated by  $d \log(\mu_p)$ , one has  $T \subset (p^{-1}O_{\bar{K}}/O_{\bar{K}})(1) = O_{\bar{K}} \otimes \mu_p$ . Since  $\Omega_{O_F}$  is isomorphic to  $O_F/p^{1-\frac{1}{p-1}}O_F$ , one has  $T = (p^{-\frac{1}{p-1}}O_{\bar{K}}/O_{\bar{K}})(1)$ .

(iv) It remains to prove surjectivity of  $(\bar{K}/O_{\bar{K}})(1) \rightarrow \Omega_{O_{\bar{K}}}$ . Let  $K' \subset \bar{K}$  be any finite extension of  $K$ ; we want to check that  $\Omega_{O_{K'}} \subset \Omega_{O_{\bar{K}}}$  lies in  $O_{\bar{K}} d \log(\mu_{p^\infty})$ . Suppose  $p^n$  kills  $\Omega_{O_{K'}}$ . Let us show that  $\Omega_{O_{K'}} \subset O_{\bar{K}} d \log(\mu_{p^{n+1}})$ . Set  $K'' := K'(\mu_{p^{n+1}})$ . The set of  $O_{K''}$ -submodules of  $\Omega_{O_{K''}}$  is totally ordered. Thus, since  $p^n d \log(\mu_{p^{n+1}}) \neq 0$  by (iii),  $\Omega_{O_{K'}}$  lies in  $O_{K''} d \log(\mu_{p^{n+1}})$ , q.e.d.  $\square$

1.4. For a complex  $P$  acyclic in degrees  $\neq 0$ , we often write  $P$  instead of  $H^0 P$ .

Consider the filtered commutative dg  $O_K$ -algebra  $A_{\text{dR}} = A_{\text{dR}} \bar{K}/K := L\Omega_{\hat{O}_{\bar{K}}/O_K}$  and the corresponding filtered  $E_\infty$   $O_K$ -algebra  $A_{\text{dR}} \hat{\otimes} \mathbb{Z}_p$  (see 1.1). Let us describe the graded  $O_{\bar{K}}$ -algebras  $\text{gr}_F^i A_{\text{dR}}, \text{gr}_F^i A_{\text{dR}} \hat{\otimes} \mathbb{Z}_p$ .

**Proposition.** (i) The complexes  $\text{gr}_F^i A_{\text{dR}} \hat{\otimes} \mathbb{Z}_p$  are acyclic in nonzero degrees, and there is a canonical isomorphism of graded algebras

$$(1.4.1) \quad \text{gr}_F^i A_{\text{dR}} \hat{\otimes} \mathbb{Z}_p \xrightarrow{\sim} \hat{O}_{\bar{K}} \langle \hat{\mathfrak{a}}(1) \rangle.$$

(ii) One has  $\text{gr}_F^0 A_{\text{dR}} = A_{\text{dR}}/F^1 = O_{\bar{K}}$ , and the complexes  $\text{gr}_F^i A_{\text{dR}}$  for  $i > 0$  are acyclic in degrees  $\neq 1$ . There are natural isomorphisms of  $O_{\bar{K}}$ -modules

$$(1.4.2) \quad \Omega^{\langle i \rangle} := H^1 \text{gr}_F^i A_{\text{dR}} \xrightarrow{\sim} (\bar{K}/i!^{-1} \mathfrak{a}^i)(i) = (\mathbb{Q}_p/\mathbb{Z}_p) \otimes i!^{-1} \hat{\mathfrak{a}}^i(i).$$

*Proof.* (i) By the theorem in 1.3, one has  $L_{O_{\bar{K}}/O_K} \xrightarrow{\sim} \Omega_{O_{\bar{K}}} \xrightarrow{\sim} (\bar{K}/\mathfrak{a})(1) = (\mathbb{Q}_p/\mathbb{Z}_p) \otimes \mathfrak{a}(1)$ ; hence  $\text{gr}_F^i A_{\text{dR}} \xrightarrow{\sim} L\Lambda_{\mathbb{Z}}^i(\mathbb{Q}_p/\mathbb{Z}_p)[-i] \otimes \mathfrak{a}^i(i)$ . One has  $L\Lambda_{\mathbb{Z}}^i(\mathbb{Q}_p/\mathbb{Z}_p) \otimes^L (\mathbb{Z}/p^n) = L\Lambda_{\mathbb{Z}/p^n}^i((\mathbb{Q}_p/\mathbb{Z}_p) \otimes^L (\mathbb{Z}/p^n)) = L\Lambda_{\mathbb{Z}/p^n}^i((\mathbb{Z}/p^n)[1])$ , which identifies with  $i!^{-1}(\mathbb{Z}/p^n)[i]$  in a way compatible with the product by (1.2.1). Therefore  $\text{gr}_F^i A_{\text{dR}} \otimes^L (\mathbb{Z}/p^n) \xrightarrow{\sim} \mathbb{Z}/p^n \langle (\mathfrak{a}/p^n \mathfrak{a})(1) \rangle$ , which yields (1.4.1).

(ii) follows from (i) by the next observation (applied to  $C = \text{gr}_F^i A_{\text{dR}}$ , with (1.4.2) defined by the condition that  $T_p(1.4.2) = (1.4.1)$ ): If a complex  $C$  of abelian groups has  $p$ -torsion cohomology and  $H^{\neq 0}(C \otimes^L \mathbb{Z}/p) = 0$ , then  $H^1 C$  is  $p$ -divisible and  $H^{\neq 1} C = 0$ .<sup>5</sup>  $\square$

1.5. By 1.4(i), the algebras  $(A_{\text{dR}}/F^i) \hat{\otimes} \mathbb{Z}_p$ , hence  $(A_{\text{dR}}/F^i) \hat{\otimes} \mathbb{Q}_p$ , are acyclic in non-zero degree. By loc.cit.,  $(A_{\text{dR}}/F^{i+1}) \hat{\otimes} \mathbb{Q}_p$  is an  $i$ -truncated dvr with residue field  $\mathbb{C}_p := \hat{O}_{\bar{K}} \otimes \mathbb{Q}$ , so  $A_{\text{dR}} \hat{\otimes} \mathbb{Q}_p := \varprojlim (A_{\text{dR}}/F^i) \hat{\otimes} \mathbb{Q}_p$  is a dvr. Let  $\mathfrak{m}_{\text{dR}}$  be its maximal ideal; (1.4.1) yields a canonical identification  $\mathfrak{m}_{\text{dR}}/\mathfrak{m}_{\text{dR}}^2 = \text{gr}_F^1 A_{\text{dR}} \hat{\otimes} \mathbb{Q}_p \xrightarrow{\sim} \mathbb{C}_p(1)$ .

<sup>5</sup>Use the fact that every complex of abelian groups splits, i.e., is quasi-isomorphic to a complex with zero differential.

**Proposition.** *There is a canonical ring isomorphism of filtered rings*

$$(1.5.1) \quad u_{\mathbb{Q}} : B_{\mathrm{dR}}^+ \xrightarrow{\sim} A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p.$$

*Proof.* The ring  $(A_{\mathrm{dR}}/F^{i+1}) \widehat{\otimes} \mathbb{Z}_p$  is an infinitesimal  $p$ -adic  $O_K$ -thickening of  $\hat{O}_K = (A_{\mathrm{dR}}/F^1) \widehat{\otimes} \mathbb{Z}_p$  of order  $\leq i$  (see [F3] 1.1). Let  $A_{\mathrm{inf}}/F^{i+1}$  be the universal thickening ([F3] 1.3); we have a canonical map  $u_i : A_{\mathrm{inf}}/F^{i+1} \rightarrow (A_{\mathrm{dR}}/F^{i+1}) \widehat{\otimes} \mathbb{Z}_p$ . Since  $B_{\mathrm{dR}}^+/F^{i+1} := (A_{\mathrm{inf}}/F^{i+1}) \otimes \mathbb{Q}$  is an  $i$ -truncated dvr and  $u_1$  is an isomorphism by [F3] 1.4.3,  $u_{i\mathbb{Q}} : B_{\mathrm{dR}}^+/F^{i+1} \xrightarrow{\sim} (A_{\mathrm{dR}}/F^{i+1}) \widehat{\otimes} \mathbb{Q}_p$ . Set  $u_{\mathbb{Q}} := \varprojlim u_{i\mathbb{Q}}$ .  $\square$

*Remarks.* (i) The map  $A_{\mathrm{dR}} \rightarrow A_{\mathrm{dR}}/F^1 = O_{\bar{K}}$  yields an isomorphism  $A_{\mathrm{dR}} \otimes \mathbb{Q} \xrightarrow{\sim} \bar{K}$ . Thus the morphism  $A_{\mathrm{dR}} \otimes \mathbb{Q} \rightarrow A_{\mathrm{dR}} \widehat{\otimes} \mathbb{Q}_p$  equals the usual embedding  $\bar{K} \hookrightarrow B_{\mathrm{dR}}^+$ . (ii) For a finite extension  $K'/K$ ,  $K' \subset \bar{K}$ , the evident map  $A_{\mathrm{dR}} \bar{K}/K \rightarrow A_{\mathrm{dR}} \bar{K}/K'$  yields an isomorphism  $A_{\mathrm{dR}} \bar{K}/K \widehat{\otimes} \mathbb{Q}_p \xrightarrow{\sim} A_{\mathrm{dR}} \bar{K}/K' \widehat{\otimes} \mathbb{Q}_p$  compatible with (1.5.1).

1.6. The next result, which will not be used in the rest of the article, is a reinterpretation of Colmez's theorem [Col]. It would be nice to find a simpler direct proof.

**Proposition.** *The complexes  $A_{\mathrm{dR}}/F^i$  are acyclic in nonzero degrees; the maps  $H^0(A_{\mathrm{dR}}/F^{i+1}) \rightarrow H^0(A_{\mathrm{dR}}/F^i)$  are injective. Set  $O^{(i)} := H^0(A_{\mathrm{dR}}/F^{i+1})$ ; thus  $O_{\bar{K}} = O^{(0)} \supset O^{(1)} \supset \dots$  and  $(A_{\mathrm{dR}}/F^{i+1}) \widehat{\otimes} \mathbb{Z}_p$  is equal to the  $p$ -adic completion  $\hat{O}^{(i)}$  of  $O^{(i)}$ .*

*Proof.* By 1.4(ii), the exact cohomology sequence for  $0 \rightarrow \mathrm{gr}_F^i A_{\mathrm{dR}} \rightarrow A_{\mathrm{dR}}/F^{i+1} \rightarrow A_{\mathrm{dR}}/F^i \rightarrow 0$  reduces to  $0 \rightarrow O^{(i)} \rightarrow O^{(i-1)} \xrightarrow{d^{(i)}} \Omega^{(i)} \rightarrow H^1(A_{\mathrm{dR}}/F^{i+1}) \rightarrow H^1(A_{\mathrm{dR}}/F^i) \rightarrow 0$ . So  $O_{\bar{K}} = O^{(0)} \supset O^{(1)} \supset \dots$ , and the vanishing of  $H^1(A_{\mathrm{dR}}/F^{i+1})$  amounts to that of  $H^1(A_{\mathrm{dR}}/F^i)$  combined with surjectivity of  $d^{(i)} : O^{(i-1)} \rightarrow \Omega^{(i)}$ . It remains to prove that all  $d^{(i)}$  are surjective.

Recall that Colmez [Col] considers a sequence of subalgebras  $O_{\bar{K}} = O^{(0)} \supset O^{(1)} \supset \dots$  and derivations  $d^{(i)} : O^{(i-1)} \rightarrow \Omega^{(i)}$  defined by induction:  $d^{(i)}$  is a universal  $O_K$ -linear derivation with values in an  $O_{\bar{K}}$ -module, and  $O^{(i)} := \mathrm{Ker} d^{(i)}$ . An induction by  $i$  shows that  $O^{(i)} \supset O^{(i)}$ : Indeed,  $\Omega^{(i)}$  are  $O_{\bar{K}}$ -modules and  $d^{(i)} : O^{(i-1)} \rightarrow \Omega^{(i)}$  is a derivation; so, if  $O^{(i-1)} \supset O^{(i-1)}$ , then  $d^{(i)}|_{O^{(i-1)}} = a^{(i)} d^{(i)}$  for some  $O_{\bar{K}}$ -linear map  $a^{(i)} : \Omega^{(i)} \rightarrow \Omega^{(i)}$ ; thus  $O^{(i)} \supset O^{(i)}$ .

Let  $i$  be the smallest number such that  $d^{(i)}$  is not surjective. Since  $E := \Omega^{(i)}/d^{(i)}(O^{(i-1)})$  is  $p$ -torsion  $p$ -divisible, one has  $E \widehat{\otimes} \mathbb{Q}_p = T_p E \otimes \mathbb{Q} \neq 0$ . Applying  $\cdot \widehat{\otimes} \mathbb{Z}_p$  to the exact triangle  $O^{(i)} \rightarrow A_{\mathrm{dR}}/F^{i+1} \rightarrow E[-1]$ , we get a short exact sequence  $0 \rightarrow \hat{O}^{(i)} \rightarrow (A_{\mathrm{dR}}/F^{i+1}) \widehat{\otimes} \mathbb{Z}_p \rightarrow T_p E \rightarrow 0$ . By [Col],  $A_{\mathrm{inf}}/F^{i+1} = \hat{O}^{(i)}$ . By universality, the map  $u_i : A_{\mathrm{inf}}/F^{i+1} \rightarrow (A_{\mathrm{dR}}/F^{i+1}) \widehat{\otimes} \mathbb{Z}_p$  equals the composition  $\hat{O}^{(i)} \rightarrow \hat{O}^{(i)} \hookrightarrow (A_{\mathrm{dR}}/F^{i+1}) \widehat{\otimes} \mathbb{Z}_p$ , so its composition with the projection onto  $T_p E$  vanishes. This cannot happen since  $u_{i\mathbb{Q}}$  is an isomorphism (see 1.5), q.e.d.  $\square$

## 2. $h$ -TOPOLOGY AND SEMI-STABLE COMPACTIFICATIONS.

2.1. *A topological digression.* The next proposition is a generalization of [V2] 4.1.

Let  $\mathcal{V}$  be an essentially small site. As in [V1], we denote by  $\mathcal{V}^\sim$  the corresponding topos (the category of sheaves of sets on  $\mathcal{V}$ ).

For us, a *base for  $\mathcal{V}$*  is a pair  $(\mathcal{B}, \phi)$ , where  $\mathcal{B}$  is an essentially small category and  $\phi : \mathcal{B} \rightarrow \mathcal{V}$  is a *faithful* functor, that satisfies the next property:

(\*) For any  $V \in \mathcal{V}$  and a finite family of pairs  $(B_\alpha, f_\alpha)$ ,  $B_\alpha \in \mathcal{B}$ ,  $f_\alpha : V \rightarrow \phi(B_\alpha)$ , there exists a set of objects  $B'_\beta \in \mathcal{B}$  and a covering family  $\{\phi(B'_\beta) \rightarrow V\}$  such that every composition  $\phi(B'_\beta) \rightarrow V \rightarrow \phi(B_\alpha)$  lies in  $\text{Hom}(B'_\beta, B_\alpha) \subset \text{Hom}(\phi(B'_\beta), \phi(B_\alpha))$ .

*Remarks.* (i) Property (\*) for empty set of  $(B_\alpha, f_\alpha)$ 's means that every  $V \in \mathcal{V}$  has a covering by objects  $\phi(B)$ ,  $B \in \mathcal{B}$ . If  $\phi$  is fully faithful, then (\*) amounts to this assertion.<sup>6</sup>

(ii) If  $\mathcal{B}$  admits finite products and  $\phi$  commutes with finite products, then it suffices to check (\*) for families  $(B_\alpha, f_\alpha)$  having  $\leq 1$  element.

(iii) In the general case, it suffices to check (\*) for families  $(B_\alpha, f_\alpha)$  having  $\leq 2$  elements.

Suppose  $(\mathcal{B}, \phi)$  is a base for  $\mathcal{V}$ . Define a covering sieve in  $\mathcal{B}$  as a sieve whose  $\phi$ -image is a covering family in  $\mathcal{V}$ .

**Proposition.** (i) Covering sieves in  $\mathcal{B}$  form a Grothendieck topology on  $\mathcal{B}$ .

(ii) The functor  $\phi : \mathcal{B} \rightarrow \mathcal{V}$  is continuous (see [V2] 1.1).

(iii)  $\phi$  yields an equivalence of the toposes: one has  $\mathcal{B}^\sim \xrightarrow{\sim} \mathcal{V}^\sim$ .

We call the above topology on  $\mathcal{B}$  the  $\phi$ -induced topology.<sup>7</sup>

*Proof.* (i) Let us check that covering sieves in  $\mathcal{B}$  are stable with respect to pullback; the rest of the axioms from [V1] 1.1 are evident. For a morphism  $g : B' \rightarrow B$  in  $\mathcal{B}$  and a covering sieve  $\mathfrak{s}$  on  $B$ , let us find a covering family on  $B'$  that belongs to the  $g$ -pullback of  $\mathfrak{s}$ . The  $\phi(g)$ -pullback of  $\phi(\mathfrak{s})$  is a covering sieve in  $\mathcal{V}$ , so there is a covering family  $\{\pi_\gamma : V_\gamma \rightarrow \phi(B')\}$  such that every composition  $V_\gamma \rightarrow \phi(B') \rightarrow \phi(B)$  can be factored as  $V_\gamma \xrightarrow{g_\gamma} \phi(B_\gamma) \xrightarrow{\phi(p_\gamma)} \phi(B)$ , where  $p_\gamma : B_\gamma \rightarrow B$  belong to  $\mathfrak{s}$ . Applying (\*) to  $V_\gamma$  and  $(B', \pi_\gamma)$ ,  $(B_\gamma, g_\gamma)$ , we find a covering family  $\{\phi(B'_{\beta_\gamma}) \rightarrow V_\gamma\}$  as in (\*). The composite covering  $\{\phi(B'_{\beta_\gamma}) \rightarrow \phi(B')\}$  comes then from a covering family  $\{B'_{\beta_\gamma} \rightarrow B'\}$  in  $\mathcal{B}$  which lies in the  $g$ -pullback of  $\mathfrak{s}$ .

(ii) We know that  $\phi$  sends covering families to covering families, so it suffices to show that for any given  $p_\alpha : B_\alpha \rightarrow B$  in  $\mathcal{B}$  and  $f_\alpha : V \rightarrow \phi(B_\alpha)$ ,  $\alpha = 1, 2$ , such that  $\phi(p_1)f_1 = \phi(p_2)f_2$  there is a covering  $\{\pi_\beta : V_\beta \rightarrow V\}$  and morphisms  $\xi_{\alpha\beta} : B'_\beta \rightarrow B_\alpha$ ,  $g_\beta : V_\beta \rightarrow \phi(B'_\beta)$  such that  $p_1\xi_{1\beta} = p_2\xi_{2\beta}$  and  $\phi(\xi_{\alpha\beta})g_\beta = f_\alpha\pi_\beta$ . Such a datum (with  $g_\beta$  the identity map) comes from (\*) applied to  $V$  and  $(B_1, f_1), (B_2, f_2)$ .

(iii) By (ii), one has the usual adjoint functors between the categories of sheaves  $(\phi^s, \phi_s) : \mathcal{B}^\sim \rightleftarrows \mathcal{V}^\sim$ . To prove that they are mutually inverse equivalences, we will check that for  $\mathcal{F} \in \mathcal{B}^\sim$  and  $\mathcal{G} \in \mathcal{V}^\sim$  the adjunction maps  $a_{\mathcal{F}} : \mathcal{F} \rightarrow \phi_s\phi^s\mathcal{F}$ ,  $b_{\mathcal{G}} : \phi^s\phi_s\mathcal{G} \rightarrow \mathcal{G}$  are isomorphisms.

Recall that  $\phi^s\mathcal{F} = (\phi^*\mathcal{F})^\sim$ , where  $\phi^*$  is the pullback of presheaves and  $\sim$  is the sheafification functor. For  $V \in \mathcal{V}$  one has  $(\phi^*\mathcal{F})(V) = \text{colim}_{\mathcal{C}(V)} \mathcal{F}$ , where  $\mathcal{C}(V)$  is the category of pairs  $(B, f)$ ,  $B \in \mathcal{B}$ ,  $f : V \rightarrow \phi(B)$ , with  $\text{Hom}_{\mathcal{C}(V)}((B, f), (B', f')) := \{g \in \text{Hom}(B', B) : \phi(g)f' = f\}$ , and we set  $\mathcal{F}(B, f) := \mathcal{F}(B)$ .

(a) To show that  $a_{\mathcal{F}}$  is an isomorphism, we check that it is injective and surjective:

$a_{\mathcal{F}}$  is injective: Suppose we have  $B \in \mathcal{B}$  and  $\xi_1, \xi_2 \in \mathcal{F}(B)$  such that  $a_{\mathcal{F}}(\xi_1) = a_{\mathcal{F}}(\xi_2)$ ; let us show that  $\xi_i$  coincide. One has  $a_{\mathcal{F}}(\xi_i) \in (\phi_s\phi^s\mathcal{F})(B) = (\phi^s\mathcal{F})(\phi(B))$ ,

<sup>6</sup>The proposition below in this situation amounts to [V2] 4.1.

<sup>7</sup>The terminology is compatible with that of [V2] 3.1.

and the equality means that there is a covering  $\{\pi_\gamma : V_\gamma \rightarrow \phi(B)\}$  such that the images of  $\xi_i$  in  $(\phi^* \mathcal{F})(V_\gamma) = \text{colim}_{\mathcal{C}(V_\gamma)} \mathcal{F}$  coincide. Thus for some *finite* subdiagram  $\mathcal{C}(V_\gamma)' \subset \mathcal{C}(V_\gamma)$  that contains  $(B, \pi_\gamma)$  the images of  $\xi_i$  in  $\text{colim}_{\mathcal{C}(V_\gamma)'} \mathcal{F}$  coincide. Applying  $(*)$  to  $V_\gamma$  and pairs from  $\mathcal{C}(V_\gamma)'$ , we get a covering  $\{\phi(B'_{\beta_\gamma}) \rightarrow V_\gamma\}$  such that the image of  $\mathcal{C}(V_\gamma)'$  in each  $\mathcal{C}(\phi(B'_{\beta_\gamma}))$  comes from a diagram in  $\mathcal{B}^\circ/B'_{\beta_\gamma}$ . The composite covering  $\{\phi(B'_{\beta_\gamma}) \rightarrow \phi(B)\}$  comes then from a covering  $\{B'_{\beta_\gamma} \rightarrow B\}$  in  $\mathcal{B}$ , and the images of  $\xi_i$  in  $\mathcal{F}(B'_{\beta_\gamma})$  coincide. Then  $\xi_1 = \xi_2$  since  $\mathcal{F}$  is a sheaf, q.e.d.

*a<sub>F</sub> is surjective:* For  $B \in \mathcal{B}$ ,  $\chi \in (\phi_s \phi^s \mathcal{F})(B)$  we look for a covering  $\{B'_\beta \rightarrow B\}$  in  $\mathcal{B}$  such that  $\chi|_{B'_\beta}$  lies in the image of  $\mathcal{F}(B'_\beta) \rightarrow (\phi_s \phi^s \mathcal{F})(B'_\beta)$ . To find it, consider  $\chi$  as an element of  $(\phi^s \mathcal{F})(\phi(B))$ . There is a covering  $\{\pi_\gamma : V_\gamma \rightarrow \phi(B)\}$  such that  $\chi|_{V_\gamma}$  lies in the image of  $(\phi^* \mathcal{F})(V_\gamma) \rightarrow (\phi^s \mathcal{F})(V_\gamma)$ , i.e., one has  $f_\gamma : V_\gamma \rightarrow \phi(B_\gamma)$  such that  $\chi|_{V_\gamma}$  lies in the image of the composition  $\mathcal{F}(B_\gamma) \rightarrow (\phi^s \mathcal{F})(\phi(B_\gamma)) \rightarrow (\phi^s \mathcal{F})(V_\gamma)$ , the second arrow comes from  $f_\gamma$ . Applying  $(*)$  to  $V_\gamma$  and  $(B, \pi_\gamma), (B_\gamma, f_\gamma)$ , we find a covering  $\{\phi(B'_{\beta_\gamma}) \rightarrow V_\gamma\}$  as in  $(*)$ ; the composite covering  $\{\phi(B'_{\beta_\gamma}) \rightarrow \phi(B)\}$  comes then from a covering  $\{B'_{\beta_\gamma} \rightarrow B\}$  that satisfies the promised property.

(b) *b<sub>G</sub> is an isomorphism:* Since  $\phi_s(b_G)a_{\phi_s \mathcal{G}} = \text{id}_{\phi_s \mathcal{G}}$  and we already know that  $a_{\phi_s \mathcal{G}}$  is an isomorphism, we see that  $\phi_s(b_G) : \phi_s \phi^s \phi_s(\mathcal{G}) \rightarrow \phi_s \mathcal{G}$  is an isomorphism. Thus  $b_G(B) : \phi^s \phi_s \mathcal{G}(\phi(B)) \rightarrow \mathcal{G}(\phi(B))$  is an isomorphism for every  $B \in \mathcal{B}$ . Since every  $V \in \mathcal{V}$  admits a covering by objects  $\phi(B)$ ,  $B \in \mathcal{B}$ , this implies that  $b_G$  is both injective and surjective, hence an isomorphism, q.e.d.  $\square$

*Exercises.* (i) For any presheaf  $\mathcal{J}$  on  $\mathcal{V}$  one has  $\phi_s(\mathcal{J}^\sim) = (\phi_* \mathcal{J})^\sim$ .

(ii) Suppose  $(\mathcal{B}, \phi)$  is a base for  $\mathcal{V}$  and  $(\mathcal{B}', \phi')$  is a base for the  $\phi$ -induced topology on  $\mathcal{B}$ . Then  $(\mathcal{B}', \phi \phi')$  is a base for  $\mathcal{V}$ .

2.2. For a field  $K$ , let  $\text{Var}_K$  be the category of  $K$ -varieties, i.e., reduced separated  $K$ -schemes of finite type. We will consider categories  $\mathcal{B}$  formed by varieties equipped with appropriate compactifications, referred to as *pairs*:

(a) *Geometric setting:* Let  $j : U \hookrightarrow \bar{U}$  be an open embedding such that  $\bar{U}$  is proper and  $U$  is dense in  $\bar{U}$ . We call such a datum *geometric pair over  $K$* , or *geometric  $K$ -pair*, and denote it by  $(U, \bar{U})$ . We say that  $(U, \bar{U})$  is a regular normal crossings pair, *nc-pair* for short, if  $\bar{U}$  is a regular scheme and  $\bar{U} \setminus U$  is a divisor with normal crossings in  $\bar{U}$ ; it is a *strict nc-pair* if the irreducible components of  $\bar{U} \setminus U$  are regular. A morphism  $f : (U', \bar{U}') \rightarrow (U, \bar{U})$  of pairs is a map  $\bar{U}' \rightarrow \bar{U}$  which sends  $U'$  to  $U$ . We denote the category of geometric  $K$ -pairs by  $\text{Var}_K^c$ ; let  $\text{Var}_K^{\text{nc}}$  be the full subcategory of nc-pairs.

(b) *Arithmetic  $K$ -setting:* Suppose  $K$  is a  $p$ -adic field as in 1.3. An *arithmetic pair over  $K$* , a.k.a. *arithmetic  $K$ -pair*, is an open embedding  $j : U \hookrightarrow \bar{U}$  with dense image of a  $K$ -variety  $U$  into a reduced proper flat  $O_K$ -scheme  $\bar{U}$ .

For such a  $(U, \bar{U})$  we set  $O_{K_U} := \Gamma(\bar{U}, \mathcal{O}_{\bar{U}})$ ,  $K_U := \Gamma(\bar{U}_K, \mathcal{O}_{\bar{U}})$ . Then  $K_U$  is the product of several finite extensions of  $K$  (labeled by the connected components of  $\bar{U}_K$ ; if  $\bar{U}$  is normal, then  $O_{K_U}$  is the product of the corresponding rings of integers. The closed fiber  $\bar{U}_s$  of  $\bar{U}$  is the union of fibers over the closed points of  $O_{K_U}$ .

We say that  $(U, \bar{U})$  is a *semi-stable pair*, or simply *ss-pair*, if (i)  $\bar{U}$  is a regular scheme, (ii)  $\bar{U} \setminus U$  is a divisor with normal crossings on  $\bar{U}$ , and (iii) the closed fiber  $\bar{U}_s$  is reduced. Our ss-pair is *strict* if the irreducible components of  $\bar{U} \setminus U$  are regular. Arithmetic  $K$ -pairs form a category  $\text{Var}_K^{\text{cc}}$ ; let  $\text{Var}_K^{\text{ss}}$  be the full subcategory of ss-pairs.

(c) *Arithmetic  $\bar{K}$ -setting:* For  $K$  as in (b), let  $\bar{K}$  be its algebraic closure. An



*arithmetic pair over  $\bar{K}$* , a.k.a. *arithmetic  $\bar{K}$ -pair*, is an open embedding  $j : V \hookrightarrow \bar{V}$  with dense image of a  $\bar{K}$ -variety  $V$  into a reduced proper flat  $O_{\bar{K}}$ -scheme  $\bar{V}$ . A connected  $(V, \bar{V})$  is said to be *semi-stable*, a.k.a. *ss-pair*, if there exists an ss-pair  $(U, \bar{U})$  over  $K$  and a  $\bar{K}$ -point  $\alpha : K_U \rightarrow \bar{K}$  (see (b)) such that  $(V, \bar{V})$  is isomorphic to  $(U, \bar{U})_\alpha = (U_{\bar{K}}, \bar{U}_{O_{\bar{K}}}) := (U \otimes_{K_U} \bar{K}, \bar{U} \otimes_{O_{K_U}} O_{\bar{K}})$ . Then  $\bar{V}$  is normal (say, by Serre's criterion). An arbitrary  $(V, \bar{V})$  is semi-stable if such are all its connected components. Denote by  $\mathcal{V}ar_K^{cc}$  the category of all arithmetic pairs over  $\bar{K}$ , and by  $\mathcal{V}ar_K^{ss} = \mathcal{V}ar_{K/K}^{ss}$  its full subcategory of ss-pairs.

*Remark.* If  $K'$  is a finite extension of  $K$  contained in  $\bar{K}$ , then  $\mathcal{V}ar_{K/K'}^{ss} \subset \mathcal{V}ar_{K/K}^{ss}$ . For all the constructions below the difference between them is irrelevant.

These categories are connected by commutative diagrams of functors

$$(2.2.1) \quad \begin{array}{ccccc} \mathcal{V}ar_K^{cc} & \rightarrow & \mathcal{V}ar_K^c & \rightarrow & \mathcal{V}ar_K, \\ \uparrow & & \uparrow & & \\ \mathcal{V}ar_K^{ss} & \rightarrow & \mathcal{V}ar_K^{nc} & & \end{array} \quad \begin{array}{ccccc} \mathcal{V}ar_{\bar{K}}^{cc} & \rightarrow & \mathcal{V}ar_{\bar{K}}^c & \rightarrow & \mathcal{V}ar_{\bar{K}}, \\ \uparrow & & \uparrow & & \\ \mathcal{V}ar_{\bar{K}}^{ss} & \rightarrow & \mathcal{V}ar_{\bar{K}}^{nc} & & \end{array}$$

where the vertical arrows are the fully faithful embeddings, and the upper horizontal lines are faithful forgetful functors of passing to the generic fiber and  $(U, \bar{U}) \mapsto U$ . The  $K$ - and  $\bar{K}$ -settings are connected by base change functors

$$(2.2.2) \quad \begin{array}{ccccc} \mathcal{V}ar_{\bar{K}}^{ss} & \rightarrow & \mathcal{V}ar_{\bar{K}}^{nc} & \rightarrow & \mathcal{V}ar_{\bar{K}} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{V}ar_K^{ss} & \rightarrow & \mathcal{V}ar_K^{nc} & \rightarrow & \mathcal{V}ar_K. \end{array}$$

Here the two right vertical arrows are the evident base change  $\cdot \otimes_K \bar{K}$ , and the left one assigns to a semi-stable  $K$ -pair  $(U, \bar{U})$  the disjoint sum of pairs  $(U, \bar{U})_\alpha$  for all  $\bar{K}$ -points  $\alpha : K_U \rightarrow \bar{K}$ .

**2.3.** A morphism  $f : (V, \bar{V}) \rightarrow (U, \bar{U})$  of pairs in either of the settings of 2.2 is called an *alteration* (of  $(U, \bar{U})$ ) if  $f^{-1}(U) = V$ , the generic fibers of  $f$  are zero-dimensional, and their union is dense in  $V$ . In setting (a),  $f$  is a (*strict*) *nc-alteration* if  $(V, \bar{V})$  is a (strict) nc-pair; in settings (b), (c),  $f$  is a (*strict*) *ss-alteration* if  $(V, \bar{V})$  is a (strict) ss-pair.

If  $f$  is an alteration, then  $f|_V : V \rightarrow U$  is proper and surjective; the composition of alterations is an alteration.

Here is a key result of de Jong [dJ1] 4.1, 6.5:

**Theorem.** *Every geometric pair admits a strict nc-alteration. Every arithmetic pair, either over  $K$  or over  $\bar{K}$ , admits a strict ss-alteration. The alterations can be chosen so that  $\bar{V}$  is projective.*  $\square$

*Remark.* Our conventions slightly differ from de Jong's: he understands varieties to be irreducible and semi-stable  $K$ -pairs  $(U, \bar{U})$  to have property  $K_U = K$ ; his notation for  $(U, \bar{U})$  is  $(\bar{U}, Z)$ ,  $Z := \bar{U} \setminus U$ .

**2.4.** For a field  $K$ , the *h-topology* (see [SV]) on  $\mathcal{V}ar_K$  is generated by the pretopology whose coverings are finite families of maps  $\{Y_\alpha \rightarrow X\}$  such that  $Y := \coprod Y_\alpha \rightarrow X$  is a universal topological epimorphism.<sup>8</sup> It is stronger than the étale and proper topologies.<sup>9</sup> We denote the h-site by  $\mathcal{V}ar_{Kh}$ ; the h-site of  $X$  is denoted by  $X_h$ .

<sup>8</sup>This means that a subset of  $X$  is Zariski open if (and only if) its preimage in  $Y$  is open, and the same is true after any base change.

<sup>9</sup>The latter is generated by a pretopology whose coverings are proper surjective maps.

*Exercise.* Let  $f : Y \rightarrow X$  be a morphism in  $\mathcal{V}ar_K$ .

- (i)  $f$  is an h-covering if (and only if) for every irreducible curve  $C \subset X$  the base change  $Y_{\tilde{C}} \rightarrow \tilde{C}$ , where  $\tilde{C}$  is the normalization of  $C$ , is an h-covering.<sup>10</sup>
- (ii) If  $X$  is a regular curve, then  $f$  is an h-covering if (and only if) the closure of the generic fiber of  $f$  maps onto  $X$ , or, equivalently,  $f$  is a covering for the flat topology.

*Remark.* By [SV] 10.4, every h-covering is a Zariski covering locally in proper topology. Therefore (see [D], [SD], or [C]) h-coverings are morphisms of universal cohomological descent for torsion étale sheaves; if  $K = \mathbb{C}$ , then h-coverings are morphisms of universal cohomological descent for arbitrary sheaves on the classical topology. In particular, for any h-hypercovering  $Y$  of  $X$  and an abelian group  $A$  the canonical map  $R\Gamma(X_{\text{ét}}, A) \rightarrow R\Gamma(Y_{\text{ét}}, A)$  ( $:=$  the total complex of the cosimplicial system of complexes  $R\Gamma(Y_{i\text{ét}}, A)$ ) is a quasi-isomorphism if  $A$  is a torsion group. Passing to the limit, we see that  $R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \xrightarrow{\sim} R\Gamma_{\text{ét}}(Y, \mathbb{Z}_p)$ . If  $K = \mathbb{C}$ , then  $R\Gamma(X_{\text{cl}}, A) \xrightarrow{\sim} R\Gamma(Y_{\text{cl}}, A)$  for any  $A$ . Since h-topology is stronger than the étale one, we see that  $R\Gamma(X_{\text{ét}}, A) \xrightarrow{\sim} R\Gamma(X_{\text{h}}, A)$  if  $A$  is a torsion group (see [SV] 10.7 for a direct proof).

2.5. Let  $\phi$  be the forgetful functor  $(U, \bar{U}) \mapsto U$  on any of the categories  $\mathcal{V}ar^?$  in 2.2.

**Proposition.** *If  $\mathcal{V}ar^?$  is either of the categories  $\mathcal{V}ar_K^c$ ,  $\mathcal{V}ar_K^{\text{nc}}$ ,  $\mathcal{V}ar_K^{\text{cc}}$ ,  $\mathcal{V}ar_K^{\text{ss}}$ , then  $(\mathcal{V}ar^?, \phi)$  is a base for  $\mathcal{V}ar_{K\text{h}}$ . If  $\mathcal{V}ar^?$  is  $\mathcal{V}ar_K^{\text{cc}}$  or  $\mathcal{V}ar_K^{\text{ss}}$ , then  $(\mathcal{V}ar^?, \phi)$  is a base for  $\mathcal{V}ar_{K\text{h}}$ .*

*Proof.* We consider the arithmetic  $K$ -setting, leaving the two other settings for the reader. Let us show that  $(\mathcal{V}ar_K^{\text{cc}}, \phi)$  satisfies condition  $(*)$  from 2.1. Our datum is a  $K$ -variety  $V$  and a finite collection of arithmetic  $K$ -pairs  $(U_\alpha, \bar{U}_\alpha)$  and maps  $f_\alpha : V \rightarrow U_\alpha$ . We need to find an h-covering  $\pi : V' \rightarrow V$  and an arithmetic pair  $(V', \bar{V}')$  such that  $f_\alpha \pi$  extend to maps  $(V', \bar{V}') \rightarrow (U_\alpha, \bar{U}_\alpha)$ . First we find an h-covering  $V' \rightarrow V$  such that  $V'$  sits in some arithmetic  $K$ -pair  $(V', \bar{V}')$ : let  $V'/V$  be a quasi-projective modification of  $V$  provided by the Chow lemma, and take for  $\bar{V}'$  the closure of  $V'$  in a projective space.<sup>11</sup> Then take  $\bar{V}'$  to be the closure of the image of  $V'$  by the embedding  $V' \hookrightarrow \bar{V}' \times \Pi \bar{U}_\alpha$ , and we are done.

To show that  $(\mathcal{V}ar_K^{\text{ss}}, \phi)$  is a base for  $\mathcal{V}ar_{K\text{h}}$ , it suffices to check that  $(\mathcal{V}ar_K^{\text{ss}}, \iota)$ , where  $\iota$  is the embedding  $\mathcal{V}ar_K^{\text{ss}} \hookrightarrow \mathcal{V}ar_K^{\text{cc}}$ , is a base for the  $\phi$ -induced topology on  $\mathcal{V}ar_K^{\text{cc}}$  (see Exercise (ii) in 2.1). Since  $\iota$  is fully faithful, it suffices to check that for every  $(U, \bar{U}) \in \mathcal{V}ar_K^{\text{cc}}$  there exists a map  $(U', \bar{U}') \rightarrow (U, \bar{U})$  such that  $U' \rightarrow U$  is an h-covering and  $(U', \bar{U}')$  is semi-stable. Such a datum is provided by the de Jong theorem in 2.3, and we are done.  $\square$

We call the  $\phi$ -induced topology on either of the categories  $\mathcal{V}ar^?$  the *h-topology*.

*Remarks.* (i) Any h-covering of  $(U, \bar{U}) \in \mathcal{V}ar_K^{\text{ss}}$  has a refinement with terms of the same dimension as  $U$  (indeed, the same assertion in  $\mathcal{V}ar_K$  is true by [SV] 10.4; to pass to  $\mathcal{V}ar_K^{\text{ss}}$ , we apply the constructions from the proof above, and they preserve the dimension).

(ii) The proposition remains true if we replace the category of ss- or nc-pairs by its

<sup>10</sup>Hint: For an open  $V \subset Y$  its image in  $X$  is constructible (EGA IV 1.8.4), so to show that  $f(V)$  is open it suffices to check that for any curve  $C \subset X$  the intersection  $C \cap f(V)$  is open in  $C$ .

<sup>11</sup>In fact, every  $V$  sits in a  $K$ -pair due to Nagata's theorem.

subcategory of strict pairs  $(U, \bar{U})$  with projective  $\bar{U}$ .

(iii) For any functor in (2.2.1) its source is a base for the h-topology of the target, and the induced topology on the source is the h-topology.

(iv) The functors in (2.2.2) are continuous for the h-topologies.

2.6. By 2.1 and 2.5,  $\phi$  identifies h-sheaves on  $\text{Var}_K$ , resp.  $\text{Var}_{\bar{K}}$ , with h-sheaves on  $\text{Var}_K^c, \text{Var}_K^{\text{nc}}, \text{Var}_K^{\text{cc}}, \text{Var}_K^{\text{ss}}$ , resp.  $\text{Var}_{\bar{K}}^c, \text{Var}_{\bar{K}}^{\text{nc}}, \text{Var}_{\bar{K}}^{\text{cc}}, \text{Var}_{\bar{K}}^{\text{ss}}$ . Thus we have the *h-localization* functors

$$(2.6.1) \quad \mathcal{PSh}(\text{Var}_K^?) \rightarrow \text{Var}_{K\text{h}}, \quad \mathcal{PSh}(\text{Var}_{\bar{K}}^?) \rightarrow \text{Var}_{\bar{K}\text{h}}$$

which assign to any presheaf  $\mathcal{F}$  on pairs the corresponding h-sheaf  $\mathcal{F}^\sim$  viewed as an h-sheaf on varieties.

*Remark.* For any presheaf on  $\text{Var}_K^c, \text{Var}_K^{\text{cc}}$  or  $\text{Var}_{\bar{K}}^{\text{cc}}$ , its h-sheafification coincides with h-sheafification of its restriction to resp.  $\text{Var}_K^{\text{nc}}, \text{Var}_K^{\text{ss}}$  or  $\text{Var}_{\bar{K}}^{\text{ss}}$ . For a presheaf on  $\text{Var}_{\bar{K}/K}^{\text{ss}}$ , its h-sheafification is the same as h-sheafification of its restriction to  $\text{Var}_{\bar{K}/K'}^{\text{ss}}$ , where  $K' \subset \bar{K}$  is any finite extension of  $K$  (see Remark in 2.2).

### 3. THE P-ADIC PERIOD MAP.

3.1. *The derived de Rham algebra in logarithmic setting.* We refer to [K1] for log scheme basics. There are two (in general, nonequivalent) ways to define the cotangent complex for log schemes due, respectively, to Gabber and Olsson, see [Ol].<sup>12</sup> Gabber's approach ([Ol] §8) is more direct and precise;<sup>13</sup> we recall it briefly.

For a commutative ring  $A$ , a prelog structure on  $A$  is a homomorphism of monoids  $\alpha : L \rightarrow A$ , where  $L$  is a commutative integral monoid (written multiplicatively) and  $A$  is viewed as a monoid with respect to the product. Rings equipped with prelog structures form a category in an evident way; denote its objects simply by  $(A, L)$ . For a fixed  $(A, L)$ , let  $\mathcal{C}_{(A, L)}$  be the category of morphisms  $(A, L) \rightarrow (B, M)$ ; we denote such an object by  $(B, M)/(A, L)$ . Let  $\Omega_{(B, M)/(A, L)}$  be the  $B$ -module of relative Kähler log differentials: it is generated by  $\Omega_{B/A}$  and elements  $d \log m, m \in M$ , subject to relations  $d \log(m_1 m_2) = d \log m_1 + d \log m_2$ ,  $\alpha(m) d \log m = d\alpha(m)$ , and  $d \log m = 0$  if  $m$  is in the image of  $L$ . The de Rham dg algebra of relative log forms  $\Omega_{(B, M)/(A, L)}^i$  has components  $\Omega_{(B, M)/(A, L)}^i := \Lambda_B^i \Omega_{(B, M)/(A, L)}$ ; elements  $d \log m$  are degree 1 cycles. It carries the Hodge filtration  $F^n = \Omega_{(B, M)/(A, L)}^{\geq n}$ .

A pair of sets  $I, J$  yields a free object  $P_{(A, L)}[I, J]$  in  $\mathcal{C}_{(A, L)}$ : the corresponding ring is a polynomial algebra  $A[t_i, t_j]_{i \in I, j \in J}$ , the monoid is  $L \oplus \mathbb{N}[I]$ , where  $\mathbb{N}[I]$  is the free monoid generated by  $I$ , and the structure map sends the generator  $m_i$  of  $\mathbb{N}[I]$ ,  $i \in I$ , to  $t_i$ . The de Rham algebra  $\Omega_{P_{(A, L)}[I, J]/(A, L)}$  is freely generated, as a graded commutative  $A$ -algebra, by elements  $t_i, t_j$  of degree 0 and  $d \log t_i := d \log m_i, dt_j$  of degree 1, where  $i \in I, j \in J$ .

Every  $(B, M)/(A, L) \in \mathcal{C}_{(A, L)}$  admits a canonical simplicial resolution  $P = P_{(A, L)}(B, M)$ . This is a simplicial object of  $\mathcal{C}_{(A, L)}$  augmented over the object  $(B, M)/(A, L)$  and such that every  $P_i$  is a free object as above. Thus we have the simplicial dg algebra  $\Omega_{P/(A, L)}$  filtered by the Hodge filtration  $F$ . Denote by

<sup>12</sup>In all situations we will consider the two versions coincide by [Ol] 8.34.

<sup>13</sup>It produces a true complex, while Olsson's construction yields a mere compatible datum of the canonical filtration truncations.

$L\Omega_{(B,M)/(A,L)}$  the corresponding total complex,  $L\Omega_{(B,M)/(A,L)}^a = \bigoplus_{j-i=a} \Omega_{P_i/(A,L)}^j$ ; this is a filtered commutative dg algebra. Let  $L\Omega_{(\hat{B},M)/(A,L)}$  be its  $F$ -completion; as in 1.2, we understand it as mere projective system of quotients  $L\Omega_{(\hat{B},M)/(A,L)}/F^n$ . One has a natural quasi-isomorphism of graded dg algebras  $\mathrm{gr}_F^* L\Omega_{(\hat{B},M)/(A,L)} \xrightarrow{\sim} (L\Lambda_B^*(L_{(B,M)/(A,L)}))[-*]$ . Here  $L_{(B,M)/(A,L)} := \Omega_{P/(A,L)} \otimes_P B$  is the relative log cotangent complex; it is acyclic in positive degrees, and  $H^0 L_{(B,M)/(A,L)} = \Omega_{(B,M)/(A,L)}$ . The constructions are compatible with direct limits. If in the above definition we replace  $P$  by any free simplicial resolution of  $(B,M)/(A,L)$ , then the output is naturally quasi-isomorphic to  $L\Omega_{(\hat{B},M)/(A,L)}$ . The plain cotangent complex and derived de Rham algebra for  $B/A$  map naturally to logarithmic ones.

For any map  $(X, \mathcal{M}) \rightarrow (S, \mathcal{L})$  of integral log schemes, the above construction, being étale sheafified, yields the log cotangent complex  $L_{(X,\mathcal{M})/(S,\mathcal{L})}$ , the derived log de Rham algebra  $L\Omega_{(X,\mathcal{M})/(S,\mathcal{L})}$ , and its  $F$ -completion  $L\Omega_{(\hat{X},\mathcal{M})/(S,\mathcal{L})}$ , which are complexes of sheaves on  $X_{\text{ét}}$ . We use only the completed complex  $L\Omega^\wedge$ .

3.2. Let  $(U, \bar{U})$  be a pair as in 2.2. We view  $\bar{U}$  as a log scheme with the usual integral log structure  $\mathcal{O}_{\bar{U}} \cap j_* \mathcal{O}_U^\times \rightarrow \mathcal{O}_{\bar{U}}$ ; by abuse of notation, let us denote this log scheme again by  $(U, \bar{U})$ . Any morphism of pairs  $(U, \bar{U}) \rightarrow (V, \bar{V})$  is a morphism of log schemes, so we have the relative log cotangent complex  $L_{(U,\bar{U})/(V,\bar{V})}$ , the derived log de Rham algebra  $L\Omega_{(\hat{U},\bar{U})/(V,\bar{V})}$ , etc., as above. There is a canonical morphism  $L\Omega_{\hat{U}/\bar{V}} \rightarrow L\Omega_{(\hat{U},\bar{U})/(V,\bar{V})}$ . We also have “absolute” complexes: in the arithmetic  $K$ - or  $\bar{K}$ -setting, these are  $L_{(U,\bar{U})} := L_{(U,\bar{U})/O_K}$ ,  $L\Omega_{(\hat{U},\bar{U})} := L\Omega_{(\hat{U},\bar{U})/O_K}$ , where  $O_K$  is considered with the trivial log structure  $O_K^\times$ ; for the geometric  $K$ - or  $\bar{K}$ -setting, replace  $O_K$  by  $K$ , resp.  $\bar{K}$ .

*Remark.* For  $(V, \bar{V}) \in \mathrm{Var}_{\bar{K}}^{\mathrm{nc}}$  one has  $L\Omega_{(\hat{V},\bar{V})} \xrightarrow{\sim} \Omega_{(V,\bar{V})}$ . Hence for  $(U, \bar{U}) \in \mathrm{Var}_{\bar{K}}^{\mathrm{ss}}$  one has  $R\Gamma(\bar{U}, L\Omega_{(\hat{U},\bar{U})}) \otimes \mathbb{Q} \xrightarrow{\sim} R\Gamma(\bar{U}_{\bar{K}}, \Omega_{(U,\bar{U}_{\bar{K}})})$ . Ditto for pairs over  $K$ .

Consider now the arithmetic  $\bar{K}$ -pair  $\mathrm{Spec}(\bar{K}, O_{\bar{K}}) := (\mathrm{Spec} \bar{K}, \mathrm{Spec} O_{\bar{K}})$ :

**Lemma.** *The cotangent complex  $L_{(\bar{K}, O_{\bar{K}})}$  is acyclic in nonzero degrees, and the canonical map  $\Omega_{O_{\bar{K}}} \rightarrow \Omega_{(\bar{K}, O_{\bar{K}})} := H^0 L_{(\bar{K}, O_{\bar{K}})}$  is an isomorphism. Therefore the canonical map  $A_{\mathrm{dR}} := L\Omega_{\hat{O}_{\bar{K}}} \rightarrow L\Omega_{(\hat{K}, O_{\bar{K}})}$  is a filtered quasi-isomorphism.*

*Proof.* For a finite extension  $K'$  of  $K$  consider the log scheme  $\mathrm{Spec}(K', O_{K'}) := (\mathrm{Spec} K', \mathrm{Spec} O_{K'})$ . It is a log complete intersection over  $O_K$  (see [Ol] 6.8). If  $\pi$  is a generator of  $O_{K'}/O_K$ ,  $f(t)$  its minimal polynomial, then, by [Ol] 6.9,  $L_{(K', O_{K'})}$  is quasi-isomorphic to the cone of the multiplication by  $f'(\pi)$  map  $O_{K'} \rightarrow O_{K'} \subset \mathfrak{m}_{K'}^{-1}$ . Thus  $L_{(K', O_{K'})}$  is acyclic in nonzero degrees,  $\Omega_{(K', O_{K'})} := H^0 L_{(K', O_{K'})}$  is a cyclic  $O_{K'}$ -module, and the canonical map  $\Omega_{O_{K'}} \rightarrow \Omega_{(K', O_{K'})}$  is an embedding with cokernel isomorphic to the residue field  $O_{K'}/\mathfrak{m}_{K'}$ . Now pass to the inductive limit, and use the fact that  $\Omega_{O_{\bar{K}}}$  is  $p$ -divisible (see 1.3).  $\square$

3.3. Consider the presheaf  $(U, \bar{U}) \mapsto R\Gamma_{\mathrm{dR}}^{\natural}(U, \bar{U}) := R\Gamma(\bar{U}, L\Omega_{(\hat{U},\bar{U})})$  of filtered  $E_\infty$  dg  $O_K$ -algebras on  $\mathrm{Var}_{\bar{K}}^{\mathrm{ss}}$ . Denote by  $\mathcal{A}_{\mathrm{dR}}^{\natural}$  its h-sheafification (2.6.1); this is an h-sheaf of filtered  $E_\infty$   $O_K$ -algebras on  $\mathrm{Var}_{\bar{K}}$  (as above, we see it as the projective system of quotients modulo  $F^i$ ). Since  $A_{\mathrm{dR}} = \mathcal{A}_{\mathrm{dR}}^{\natural}(\mathrm{Spec} \bar{K})$  by Lemma in 3.2,  $A_{\mathrm{dR}}$ , viewed as a constant filtered h-sheaf, maps into  $\mathcal{A}_{\mathrm{dR}}^{\natural}$ .

**Theorem** (*p*-adic Poincaré lemma). *The maps  $A_{\mathrm{dR}} \otimes^L \mathbb{Z}/p^n \rightarrow \mathcal{A}_{\mathrm{dR}}^{\natural} \otimes^L \mathbb{Z}/p^n$  are filtered quasi-isomorphisms of h-sheaves on  $\mathcal{V}ar_{\bar{K}}$ .*

For a proof, see §4. Assuming it, let us define the *p*-adic period map  $\rho$ .

3.4. *The Hodge-Deligne filtration.* For this subsection,  $K$  is any field of characteristic 0. Consider the presheaf  $(V, \bar{V}) \mapsto R\Gamma_{\mathrm{dR}}(V, \bar{V}) := R\Gamma(\bar{V}, \Omega_{(V, \bar{V})})$  of filtered  $E_{\infty}$  dg  $K$ -algebras on  $\mathcal{V}ar_K^{\mathrm{nc}}$ . Let  $\mathcal{A}_{\mathrm{dR}}$  its h-sheafification (2.6.1), which is an h-sheaf of filtered  $E_{\infty}$   $K$ -algebras on  $\mathcal{V}ar_K$  (viewed as the projective system of quotients modulo  $F^i$ ). For any  $X \in \mathcal{V}ar_K$  set

$$(3.4.1) \quad R\Gamma_{\mathrm{dR}}(X) := R\Gamma(X_{\mathrm{h}}, \mathcal{A}_{\mathrm{dR}}).$$

This is Deligne's de Rham complex of  $X$  equipped with Deligne's Hodge filtration.

**Proposition.** (i) *For  $(V, \bar{V}) \in \mathcal{V}ar_K^{\mathrm{nc}}$  the canonical map  $R\Gamma_{\mathrm{dR}}(V, \bar{V}) \rightarrow R\Gamma_{\mathrm{dR}}(V)$  is a filtered quasi-isomorphism.*

(ii) *The differential of  $R\Gamma_{\mathrm{dR}}(X)$  is strictly compatible with the filtration.  $H_{\mathrm{dR}}^i(X) := H^i R\Gamma_{\mathrm{dR}}(X)$  are  $K$ -vector spaces of dimension equal to  $\dim H_{\mathrm{ét}}^i(X_{\bar{K}}, \mathbb{Q}_p)$ .*

(iii) *For any smooth variety  $X$  there is a canonical (nonfiltered) quasi-isomorphism  $R\Gamma(X, \Omega_X) \xrightarrow{\sim} R\Gamma_{\mathrm{dR}}(X)$ .*

*Proof.* By Lefschetz's principle, we can assume that  $K = \mathbb{C}$ . For  $(V, \bar{V}) \in \mathcal{V}ar_{\mathbb{C}}^{\mathrm{nc}}$  the maps  $R\Gamma_{\mathrm{dR}}(V, \bar{V}) \rightarrow R\Gamma_{\mathrm{dR}}(V, \Omega_V) \rightarrow R\Gamma(V_{\mathrm{cl}}, \mathbb{C})$  are quasi-isomorphisms by [Gr]. Thus for any h-hypercovering  $(Y, \bar{Y})/X$  of  $X$  in  $\mathcal{V}ar_K^{\mathrm{nc}}$ <sup>14</sup> the cohomological descent (see Remark in 2.4) yields a canonical quasi-isomorphism  $R\Gamma(\bar{Y}, \Omega_{(Y, \bar{Y})}) \xrightarrow{\sim} R\Gamma(X_{\mathrm{cl}}, \mathbb{C})$ . If we equip  $R\Gamma(X_{\mathrm{cl}}, \mathbb{C})$  with the Hodge-Deligne filtration of mixed Hodge theory [D], then this is a filtered quasi-isomorphism. Therefore we have a canonical filtered quasi-isomorphism  $R\Gamma_{\mathrm{dR}}(X) \xrightarrow{\sim} R\Gamma(X_{\mathrm{cl}}, \mathbb{C})$ . Now (i) and the second assertion of (ii) are clear; the first assertion of (ii) follows from mixed Hodge theory. The quasi-isomorphism in (iii) is  $R\Gamma(X, \Omega_X) \xrightarrow{\sim} R\Gamma(Y, \Omega_Y) \xleftarrow{\sim} R\Gamma(\bar{Y}, \Omega_{(Y, \bar{Y})})$ , where the arrows are quasi-isomorphisms by the cohomological descent (since  $R\Gamma(X, \Omega_X) \xrightarrow{\sim} R\Gamma(X_{\mathrm{cl}}, \mathbb{C})$ ).  $\square$

3.5. We return to the setting of 3.3, so  $K$  is our *p*-adic field. Let  $X$  be any variety over  $\bar{K}$ . It yields a filtered  $E_{\infty}$   $O_K$ -algebra

$$(3.5.1) \quad R\Gamma_{\mathrm{dR}}^{\natural}(X) := R\Gamma(X_{\mathrm{h}}, \mathcal{A}_{\mathrm{dR}}^{\natural}).$$

Since  $A_{\mathrm{dR}} \otimes \mathbb{Q} = \bar{K}$  (see Remark (i) in 1.5),  $R\Gamma_{\mathrm{dR}}^{\natural}(X) \otimes \mathbb{Q}$  is a  $\bar{K}$ -algebra. By Remark in 3.2, we have a filtered quasi-isomorphism of  $E_{\infty}$   $\bar{K}$ -algebras

$$(3.5.2) \quad R\Gamma_{\mathrm{dR}}^{\natural}(X) \otimes \mathbb{Q} \xrightarrow{\sim} R\Gamma_{\mathrm{dR}}(X).$$

Let us compute  $R\Gamma_{\mathrm{dR}}^{\natural}(X) \widehat{\otimes} \mathbb{Z}_p$ . Consider the morphisms of filtered complexes  $R\Gamma(X_{\mathrm{ét}}, \mathbb{Z}) \otimes^L A_{\mathrm{dR}} \xrightarrow{\sim} R\Gamma(X_{\mathrm{ét}}, A_{\mathrm{dR}}) \rightarrow R\Gamma(X_{\mathrm{h}}, A_{\mathrm{dR}}) \rightarrow R\Gamma(X_{\mathrm{h}}, \mathcal{A}_{\mathrm{dR}}^{\natural}) = R\Gamma_{\mathrm{dR}}^{\natural}(X)$ . After applying  $\cdot \otimes^L \mathbb{Z}/p^n$ , the arrows become filtered quasi-isomorphisms (the first one by Remark in 2.4, the second one by the Poincaré lemma in 3.3), so we get

<sup>14</sup>Here we view  $X$  as an h-sheaf on  $\mathcal{V}ar_K^{\mathrm{nc}}$ , so  $(Y, \bar{Y})$  is a simplicial object of  $\mathcal{V}ar_K^{\mathrm{nc}}$  equipped with an augmentation map  $Y \rightarrow X$  that makes  $Y$  an h-hypercovering of  $X$ .

a filtered quasi-isomorphism  $R\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n) \otimes^L A_{\text{dR}} \xrightarrow{\sim} R\Gamma_{\text{dR}}^{\natural}(X) \otimes^L \mathbb{Z}/p^n$ . Since  $R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) = \text{holim} R\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n) = R\Gamma(X_{\text{ét}}, \mathbb{Z}) \hat{\otimes} \mathbb{Z}_p$  (see 1.1) is a perfect  $\mathbb{Z}_p$ -complex and  $R\Gamma(X_{\text{ét}}, \mathbb{Z}/p^n) = R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p}^L \mathbb{Z}/p^n$ , one has, passing to the homotopy limit as in 1.1,  $R\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p}^L (A_{\text{dR}} \hat{\otimes} \mathbb{Z}_p) \xrightarrow{\sim} R\Gamma_{\text{dR}}^{\natural}(X) \hat{\otimes} \mathbb{Z}_p$ . Tensoring by  $\mathbb{Q}$ , we get a filtered quasi-isomorphism of filtered  $E_{\infty} B_{\text{dR}}^+$ -algebras (see (1.5.1))

$$(3.5.3) \quad \beta : R\Gamma_{\text{ét}}(X, \mathbb{Q}_p) \otimes B_{\text{dR}}^+ \xrightarrow{\sim} R\Gamma_{\text{dR}}^{\natural}(X) \hat{\otimes} \mathbb{Q}_p.$$

Let  $\alpha : R\Gamma_{\text{dR}}(X) \otimes_{\bar{K}} B_{\text{dR}}^+ \rightarrow R\Gamma_{\text{dR}}^{\natural}(X) \hat{\otimes} \mathbb{Q}_p$  be the  $B_{\text{dR}}^+$ -linear extension of the composition  $R\Gamma_{\text{dR}}(X) \xrightarrow{\sim} R\Gamma_{\text{dR}}^{\natural}(X) \otimes \mathbb{Q} \rightarrow R\Gamma_{\text{dR}}^{\natural}(X) \hat{\otimes} \mathbb{Q}_p$ , where the first arrow is inverse to (3.5.2) and the second one comes from the canonical map  $? \rightarrow ? \hat{\otimes} \mathbb{Z}_p$ . We get a morphism of filtered  $E_{\infty} B_{\text{dR}}^+$ -algebras

$$(3.5.4) \quad \rho = \rho_{\text{dR}} := \beta^{-1} \alpha : R\Gamma_{\text{dR}}(X) \otimes_{\bar{K}} B_{\text{dR}}^+ \rightarrow R\Gamma_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}}^+.$$

*Remarks.* (i) The Galois group  $\text{Gal}(\bar{K}/K)$  acts on  $\text{Var}_{\bar{K}h}$  and on both sides of (3.5.4) by transport of structure, and  $\rho^+$  is evidently compatible with this action. In particular, if  $X$  is defined over  $K$ , i.e.,  $X = X_K \otimes_K \bar{K}$ , then  $R\Gamma_{\text{dR}}(X) = R\Gamma_{\text{dR}}(X_K) \otimes_K \bar{K}$ , and we can rewrite (3.5.4) as a  $\text{Gal}(\bar{K}/K)$ -equivariant morphism

$$(3.5.5) \quad \rho : R\Gamma_{\text{dR}}(X_K) \otimes_K B_{\text{dR}}^+ \rightarrow R\Gamma_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}}^+.$$

(ii) The map  $\rho$  does not change if we replace  $K$  by any its finite extension that contains in  $\bar{K}$  (see Remark in 2.6).

**3.6. Theorem.** *The  $B_{\text{dR}}$ -linear extension of  $\rho$  is a filtered quasi-isomorphism: for any  $X \in \text{Var}_{\bar{K}}$  one has*

$$(3.6.1) \quad \rho : R\Gamma_{\text{dR}}(X) \otimes_{\bar{K}} B_{\text{dR}} \xrightarrow{\sim} R\Gamma_{\text{ét}}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}}.$$

*Proof.* (a) *The case of  $X = \mathbb{G}_m = \mathbb{G}_{m\bar{K}}$ :* The  $\bar{K}$ -line  $H_{\text{dR}}^1(\mathbb{G}_m) = \text{gr}_F^1 H_{\text{dR}}^1(\mathbb{G}_m)$  is generated by  $d \log t$ . The  $\mathbb{Z}_p$ -line  $H_{\text{ét}}^1(\mathbb{G}_m, \mathbb{Z}_p)(1) = H_{\text{ét}}^1(\mathbb{G}_m, \mathbb{Z}_p(1))$  is generated by the class  $cl(\mathfrak{k})$  of the Kummer  $\mathbb{Z}_p(1)$ -torsor  $\mathfrak{k} = \varprojlim \mathfrak{k}_n$ ,  $\mathfrak{k}_n := (t^{1/p^n})$ . Due to the canonical identification  $\mathbb{C}_p(1) \xrightarrow{\sim} \mathfrak{m}_{\text{dR}}/\mathfrak{m}_{\text{dR}}^2 = \text{gr}_F^1 B_{\text{dR}}$ , see 1.4, 1.5, we can view  $cl(\mathfrak{k})$  as a generator of the  $\mathbb{C}_p$ -line  $H_{\text{ét}}^1(\mathbb{G}_m, \mathbb{Q}_p) \otimes \text{gr}_F^1 B_{\text{dR}}$ .

**Lemma.** *One has  $\text{gr}_F^1(\rho)(d \log t) = cl(\mathfrak{k}) \in H_{\text{ét}}^1(\mathbb{G}_m, \mathbb{Q}_p) \otimes \text{gr}_F^1 B_{\text{dR}}$ .*

*Proof of Lemma.* We make a mod  $p^n$  computation. Consider the ss-pair  $(\mathbb{G}_m, \bar{\mathbb{G}}_m)$ ,  $\bar{\mathbb{G}}_m := \mathbb{P}_{O_{\bar{K}}}^1$ . One has  $\text{gr}_F^1 L\Omega^1(\hat{\mathbb{G}}_m, \bar{\mathbb{G}}_m) = \Omega_{(\mathbb{G}_m, \bar{\mathbb{G}}_m)}^1[-1]$ , so  $d \log t \in \Gamma(\bar{\mathbb{G}}_m, \Omega_{(\mathbb{G}_m, \bar{\mathbb{G}}_m)}^1)$  is a 1-cocycle in  $\text{gr}_F^1 R\Gamma_{\text{dR}}^{\natural}(\mathbb{G}_m)$ . As in 1.1, set  $C_n := \text{Cone}(p^n : \mathbb{Z} \rightarrow \mathbb{Z})$ . Let  $d \log cl(\mathfrak{k}_n)$  be the image of the class  $cl(\mathfrak{k}_n)$  of  $\mathfrak{k}_n$  by the composition  $H^1(\mathbb{G}_m, \mu_{p^n}) \rightarrow H^1(\mathbb{G}_m, \text{ét}, \text{gr}_F^1 A_{\text{dR}} \otimes C_n) \rightarrow \text{gr}_F^1 R\Gamma_{\text{dR}}^{\natural}(\mathbb{G}_m) \otimes C_n$ , where the first arrow comes from the coefficient maps  $\mu_{p^n} \xrightarrow{d \log} \Omega_{O_{\bar{K}}} p^n \hookrightarrow \Omega_{O_{\bar{K}}}[-1] \otimes C_n = \text{gr}_F^1 A_{\text{dR}} \otimes C_n$ . To prove the lemma, we will show that the image of  $d \log t$  by the embedding  $\text{gr}_F^1 R\Gamma_{\text{dR}}^{\natural}(\mathbb{G}_m) \hookrightarrow \text{gr}_F^1 R\Gamma_{\text{dR}}^{\natural}(\mathbb{G}_m) \otimes C_n$  is homologous to  $d \log cl(\mathfrak{k}_n)$ .

Let  $\tilde{\mathbb{G}}_m$  be a copy of  $\mathbb{G}_m$  with parameter  $\tilde{t}$ , and  $\pi : \tilde{\mathbb{G}}_m \rightarrow \mathbb{G}_m$  be the projection  $t = \tilde{t}^{p^n}$ . Thus  $\tilde{\mathbb{G}}_m/\mathbb{G}_m$  is our  $\mu_{p^n}$ -torsor  $\mathfrak{k}_n$ , so  $cl(\mathfrak{k}_n)$  is represented by a Čech  $\mu_{p^n}$ -cocycle  $c(\mathfrak{k}_n)$  for the étale covering  $\tilde{\mathbb{G}}_m/\mathbb{G}_m$ . The corresponding Čech hypercovering

is the twist of  $\mathbb{G}_m$  by the universal  $\mu_{p^n}$ -torsor  $\mathfrak{t}_n$  over the classifying simplicial space  $B_{\mu_{p^n}}$ , so for any sheaf  $\mathcal{F}$  the Čech complex of  $\mathbb{G}_m/\mathbb{G}_m$  with coefficients in  $\mathcal{F}$  is the cochain complex  $C^*(\mu_{p^n}, \Gamma(\mathbb{G}_m, \mathcal{F}))$  for  $\mu_{p^n}$  acting on sections by the translations. The 1-cocycle  $c(\mathfrak{t}_n)$  is the identity map  $\mu_{p^n} \rightarrow \mu_{p^n} = \Gamma(\mathbb{G}_m, \mu_{p^n})$ .

Our  $\pi$  extends to the h-covering of semi-stable pairs  $(\mathbb{G}_m, \bar{\mathbb{G}}_m) \rightarrow (\mathbb{G}_m, \bar{\mathbb{G}}_m)$ , and the Čech hypercovering extends to a hypercovering in  $\mathcal{V}ar_K^{\text{ss}}$  which is the  $\mathfrak{t}_n$ -twist of  $(\mathbb{G}_m, \bar{\mathbb{G}}_m)$ . So one has a canonical map  $C^*(\mu_{p^n}, \Gamma(\bar{\mathbb{G}}_m, \Omega_{(\mathbb{G}_m, \bar{\mathbb{G}}_m)}^1))[-1] \rightarrow \text{gr}_F^1 R\Gamma_{\text{dR}}^{\mathfrak{h}}(\mathbb{G}_m)$ , hence  $C^*(\mu_{p^n}, \Gamma(\bar{\mathbb{G}}_m, \Omega_{(\mathbb{G}_m, \bar{\mathbb{G}}_m)}^1))[-1] \otimes C_n \rightarrow \text{gr}_F^1 R\Gamma_{\text{dR}}^{\mathfrak{h}}(\mathbb{G}_m) \otimes C_n$ . Both  $d \log t$  and  $d \log c(\mathfrak{t}_n)$  are 1-cocycles in  $C^*(\mu_{p^n}, \Gamma(\bar{\mathbb{G}}_m, \Omega_{(\mathbb{G}_m, \bar{\mathbb{G}}_m)}^1))[-1] \otimes C_n$ : namely,  $d \log t \in C^0(\mu_{p^n}, \Gamma(\bar{\mathbb{G}}_m, \Omega_{(\mathbb{G}_m, \bar{\mathbb{G}}_m)}^1))[-1]$  and  $d \log c(\mathfrak{t}_n) \in C^1(\mu_{p^n}, \Omega_{O_{\bar{K}/K}}(\mathfrak{p}^n)) \subset C^1(\mu_{p^n}, \Gamma(\bar{\mathbb{G}}_m, \Omega_{(\mathbb{G}_m, \bar{\mathbb{G}}_m)}^1))[-1] \otimes C_n$ . Their difference is the differential of the 0-cochain  $d \log \tilde{t} \in C^0(\mu_{p^n}, \Gamma(\bar{\mathbb{G}}_m, \Omega_{(\mathbb{G}_m, \bar{\mathbb{G}}_m)}^1)) \subset C^0(\mu_{p^n}, \Gamma(\bar{\mathbb{G}}_m, \Omega_{(\mathbb{G}_m, \bar{\mathbb{G}}_m)}^1))[-1] \otimes C_n$ , q.e.d.  $\square$

We see that for  $X = \mathbb{G}_m$  the map  $\rho$  of (3.6.1) is a filtered quasi-isomorphism. It provides a canonical generator  $\rho(d \log t)/cl(\mathfrak{t})$  of  $\mathfrak{m}_{\text{dR}}(-1)$ . Thus we have a canonical identification between  $H_{\text{ét}}^i(X, \mathbb{Q}_p(n)) \otimes \mathbb{B}_{\text{dR}}$  and  $H_{\text{ét}}^i(X, \mathbb{Q}_p) \otimes \mathbb{B}_{\text{dR}}$  with the filtration shifted by  $n$ , etc.

*Remark.* The above generator is equal to Fontaine's generator, see [F3] 1.5.4. Indeed, they coincide modulo  $\mathfrak{m}_{\text{dR}}^2(-1)$  by the lemma, and both are  $\text{Gal}(\bar{K}/K)$ -invariant (see Remark in 3.5). Since  $H^0(\text{Gal}(\bar{K}/K), \mathfrak{m}_{\text{dR}}^2(-1)) = 0$ , we are done.

(b) *Compatibility with the Gysin maps:* Let  $i : Y \hookrightarrow X$  be a closed codimension 1 embedding of smooth varieties. It yields the Gysin isomorphisms  $i_{*\text{dR}} : R\Gamma_{\text{dR}}(Y) \xrightarrow{\sim} R\Gamma_{\text{dR}Y}(X)(1)[2] := \text{Cone}(R\Gamma_{\text{dR}}(X) \rightarrow \Gamma_{\text{dR}}(X \setminus Y))(1)[1]$ ,  $i_{*\mathbb{Q}_p} : R\Gamma_{\text{ét}}(Y, \mathbb{Q}_p) \xrightarrow{\sim} R\Gamma_{\text{ét}Y}(X, \mathbb{Q}_p)(1)[2]$ . Let us show that  $\rho$  commutes with the Gysin maps.

Consider the deformation to the normal cone diagram

$$(3.6.2) \quad \begin{array}{ccccc} \mathcal{L} & \hookrightarrow & X_{\mathbb{A}^1} & \hookleftarrow & X \\ \uparrow & & \uparrow & & \uparrow \\ Y & \hookrightarrow & Y_{\mathbb{A}^1} & \hookleftarrow & Y. \end{array}$$

Here  $Y_{\mathbb{A}^1} = Y \times \mathbb{A}^1$ ,  $X_{\mathbb{A}^1}$  is  $X \times \mathbb{A}^1$  with  $Y \times \{0\}$  blown up, the left arrow is the zero section of the normal bundle  $\mathcal{L}$  to  $Y$  in  $X$ , and the bottom embeddings are  $y \mapsto (y, 0), (y, 1)$ . It yields a commutative diagram of the de Rham cohomology

$$(3.6.3) \quad \begin{array}{ccccc} R\Gamma_{\text{dR}Y}(\mathcal{L})(1)[2] & \leftarrow & R\Gamma_{\text{dR}Y_{\mathbb{A}^1}}(X_{\mathbb{A}^1})(1)[2] & \rightarrow & R\Gamma_{\text{dR}Y}(X)(1)[2] \\ \uparrow & & \uparrow & & \uparrow \\ R\Gamma_{\text{dR}}(Y) & \leftarrow & R\Gamma_{\text{dR}}(Y_{\mathbb{A}^1}) & \rightarrow & R\Gamma_{\text{dR}}(Y), \end{array}$$

where the vertical arrows are the Gysin isomorphisms and the horizontal ones are pullbacks. There is a similar diagram for the  $\mathbb{Q}_p$ -cohomology. The horizontal maps are filtered quasi-isomorphisms, so, since  $\rho$  is compatible with pullbacks, we see that the Gysin compatibility for  $Y \hookrightarrow X$  amounts to one for  $Y \hookrightarrow \mathcal{L}$ .

So we can assume that  $X$  is a line bundle  $\mathcal{L}$  over  $Y$  and  $i$  its zero section. Now the source of both  $i_*$ 's are dg algebras, the targets are modules over them (due to the projection  $\mathcal{L} \rightarrow Y$ ), and  $i_*$ 's are morphisms of modules. Thus it suffices to

check that  $\rho$  identifies the images of 1. The assertion is local with respect to  $Y$ , hence we can assume that  $\mathcal{L}$  is trivial. By base change, we reduced to the case when  $Y$  is a point, where we are done by (a).

(c) *The case of a smooth projective  $X$ :* Let us check that the morphism of bigraded rings  $\mathrm{gr}_F \rho^* : \mathrm{gr}_F H_{\mathrm{dR}}^*(X) \otimes_{\bar{K}} \mathrm{gr}_F B_{\mathrm{dR}} \rightarrow H_{\mathrm{\acute{e}t}}^*(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathrm{gr}_F B_{\mathrm{dR}}$  is an isomorphism. It is an isomorphism for  $* = 0$ . By (b),  $\mathrm{gr}_F \rho^2$  identifies the classes  $c$  of a hyperplane section. Since the product with  $c^{\dim X}$  identifies  $H^0$  and  $H^{2 \dim X}$ ,  $\mathrm{gr}_F \rho^{2 \dim X}$  is an isomorphism. Therefore, since  $\mathrm{gr}_F \rho^*$  is compatible with the Poincaré pairing for classes of opposite degrees and the latter is nondegenerate,  $\mathrm{gr}_F \rho^*$  is injective. Since  $\dim_{\bar{K}} H_{\mathrm{dR}}^*(X) = \dim_{\mathbb{Q}_p} H_{\mathrm{\acute{e}t}}^*(X, \mathbb{Q}_p)$ , we are done.

(d) *The case of  $X = \bar{X} \setminus D$ , where  $\bar{X}$  is smooth projective,  $D$  is a strict normal crossings divisor:* Let  $Y$  be an irreducible component of  $D$ ,  $D'$  be the union of the other components; set  $X' := \bar{X} \setminus D'$ ,  $Y' := Y \setminus D'$ . By induction by the number of components  $D$  (starting with (c)), we can assume that the theorem holds for  $X'$  and  $Y'$ . By (b),  $\rho$  provides a morphism between the exact Gysin triangles for  $(Y', X')$ . It is a filtered quasi-isomorphism on the  $X'$  and  $Y'$  terms; hence it is a filtered quasi-isomorphism on the  $X$  term, q.e.d.

(e) *The case of arbitrary  $X$ :* If  $Y/X$  is any h-hypercovering of  $X$ , then the canonical map  $R\Gamma_{\mathrm{dR}}(X) \rightarrow R\Gamma_{\mathrm{dR}}(Y)$  (which is the total complex of the cosimplicial system of filtered complexes  $R\Gamma_{\mathrm{dR}}(Y_i)$ ) is a filtered quasi-isomorphism by the construction of  $R\Gamma_{\mathrm{dR}}$ , and  $R\Gamma_{\mathrm{\acute{e}t}}(X, \mathbb{Q}_p) \xrightarrow{\sim} R\Gamma_{\mathrm{\acute{e}t}}(Y, \mathbb{Q}_p)$  by cohomological descent (see Remark in 2.4). Thus if  $\rho$  is a filtered quasi-isomorphism for every  $Y_i$ , then it is a filtered quasi-isomorphism for  $X$ . We are done, since, by de Jong (or Hironaka), one can find  $Y/X$  with  $Y_i$  as in (d).  $\square$

*Remark.*  $\rho$  is compatible with Chern classes of vector bundles: Indeed,  $c_i(E)$  are determined in the usual way by  $c_1(\mathcal{O}(1)_{\mathbb{P}(E)})$ , so it suffices to show that  $\rho$  identifies  $c_1$ 's of line bundles. Notice that the construction of  $\rho$  extends tautologically to simplicial schemes. By (a) above,  $\rho$  identifies the de Rham and étale Chern classes of the universal line bundle over the classifying simplicial scheme  $B_{\mathbb{G}_m}$ . For a line bundle  $\mathcal{L}$  on  $X$ , choose a finite open covering  $\{U_i\}$  of  $X$  such that  $\mathcal{L}$  is trivial on  $U_i$ ; let  $\pi : X^\sim \rightarrow X$  be the Čech hypercovering. Since  $\pi$  yields an isomorphism between the cohomology, it suffices to check that  $\rho$  identifies the Chern classes of  $\pi^* \mathcal{L}$ . This is true since  $\pi^* \mathcal{L}$  is the pullback of the universal line bundle by a map  $X^\sim \rightarrow B_{\mathbb{G}_m}$ .

#### 4. PROOF OF THE POINCARÉ LEMMA.

4.1. Pick any  $(V, \bar{V}) \in \mathcal{V}ar_{\bar{K}}^{\mathrm{ss}}$ .

**Proposition.** *One has  $L_{(V, \bar{V})} \xrightarrow{\sim} \Omega_{(V, \bar{V})}$ , the  $\mathcal{O}_{\bar{V}}$ -module  $\Omega_{\langle V, \bar{V} \rangle} := \Omega_{(V, \bar{V})}/(\bar{K}, \mathcal{O}_{\bar{K}})$  is locally free of finite rank, and there is a canonical short exact sequence*

$$(4.1.1) \quad 0 \rightarrow \mathcal{O}_{\bar{V}} \otimes_{\mathcal{O}_{\bar{K}}} \Omega_{\mathcal{O}_{\bar{K}}} \rightarrow \Omega_{(V, \bar{V})} \rightarrow \Omega_{\langle V, \bar{V} \rangle} \rightarrow 0.$$

*Proof.* We can assume that  $V$  is connected, so  $(V, \bar{V})$  is the base change of a semi-stable  $K$ -pair  $(U, \bar{U})$  as in 2.2(c), i.e.,  $(V, \bar{V}) = (U_{\bar{K}}, \bar{U}_{\mathcal{O}_{\bar{K}}})$ . For any finite extension  $K'$  of  $K$ , consider an arithmetic  $K$ -pair  $(U_{K'}, \bar{U}_{\mathcal{O}_{K'}}) := (U \otimes_{K_U} K', \bar{U} \otimes_{\mathcal{O}_{K_U}} \mathcal{O}_{K'})$ . Set  $\Omega_{\langle U, \bar{U} \rangle} := \Omega_{(U, \bar{U})}/(K_U, \mathcal{O}_{K_U})$ ,  $\Omega_{\langle U_{K'}, \bar{U}_{\mathcal{O}_{K'}} \rangle} := \Omega_{(U_{K'}, \bar{U}_{\mathcal{O}_{K'}})}/(K', \mathcal{O}_{K'})$ .



**Lemma.** *The log scheme  $(U_{K'}, \bar{U}_{O_{K'}})$  coincides with the pullback of  $(U, \bar{U})$  by the map  $\text{Spec}(K', O_{K'}) \rightarrow \text{Spec}(K_U, O_{K_U})$  in the category of log schemes.*

Assume the lemma for a moment. The map  $(U, \bar{U}) \rightarrow \text{Spec}(O_{K_U}, K_U)$  is log smooth and integral; by the lemma,  $(U_{K'}, \bar{U}_{O_{K'}}) \rightarrow \text{Spec}(O_{K'}, K')$  enjoys the same properties. So, by [Ol] 8.34,  $L_{(U_{K'}, \bar{U}_{O_{K'}})/(K', O_{K'})} \xrightarrow{\sim} \Omega_{(U_{K'}, \bar{U}_{O_{K'}})} = O_{K'} \otimes_{O_{K_U}} \Omega_{(U, \bar{U})}$ , which is a locally free  $\mathcal{O}_{\bar{U}_{O_{K'}}}$ -module of finite rank. Since  $L_{(K', O_{K'})} \xrightarrow{\sim} \Omega_{(K', O_{K'})}$  (see the proof of Lemma in 3.2), the canonical exact triangle ([Ol] 8.18)  $\mathcal{O}_{\bar{U}_{O_{K'}}} \otimes_{O_{K'}} L_{(K', O_{K'})} \rightarrow L_{(U_{K'}, \bar{U}_{O_{K'}})} \rightarrow L_{(U_{K'}, \bar{U}_{O_{K'}})/(K', O_{K'})}$  reduces to the short exact sequence  $0 \rightarrow \mathcal{O}_{\bar{U}_{O_{K'}}} \otimes_{O_{K'}} \Omega_{(K', O_{K'})} \rightarrow \Omega_{(U_{K'}, \bar{U}_{O_{K'}})} \rightarrow \Omega_{(U_{K'}, \bar{U}_{O_{K'}})} \rightarrow 0$ . Pass to the limit by all  $K' \subset \bar{K}$  and use the lemma in 3.2; we are done.  $\square$

*Proof of Lemma.* The underlying scheme of the pullback log scheme is  $\bar{U}_{O_{K'}}$ . Let us show that its log structure map  $\mathcal{M} \rightarrow \mathcal{O}_{\bar{U}_{O_{K'}}} \cap j_* \mathcal{O}_{U_{K'}}^\times$  is an isomorphism. The assertion is étale local, so we can assume that  $\bar{U}$  is étale over  $\text{Spec } O_{K_U}[t_a, t_b, t_c]/(\Pi t_a - \pi_{K_U})$ , where  $a, b, c$  are in finite sets  $A, B, C$ ,  $\pi_{K_U}$  is a uniformizing parameter in  $O_{K_U}$ , and  $U$  is the subscheme where all  $t_a, t_b$  are invertible. The log structure of  $(U, \bar{U})$  is fine with a chart  $\mathbb{N}[A \sqcup B] \rightarrow \mathcal{O}_{\bar{U}}$ , which sends generators  $m_a, m_b$  of  $\mathbb{N}[A \sqcup B]$  to  $t_a, t_b$ . Therefore<sup>15</sup>  $\bar{U}_{O_{K'}}$  is étale over  $\text{Spec } O_{K'}[t_a, t_b, t_c]/(\Pi t_a - \pi_{K'}^e)$ , where  $e$  is the ramification index of  $K'/K_U$ ,  $\pi_{K'}$  is a uniformizing parameter in  $O_{K'}$ , and the log structure  $\mathcal{M}$  has a chart  $M_{A,B} \rightarrow \mathcal{O}_{\bar{U}_{O_{K'}}}$ , where  $M_{A,B}$  is the quotient of  $\mathbb{N}[A \sqcup B] \oplus \mathbb{N}$  modulo the relation  $\Pi m_a = m_\pi^e$  ( $m_\pi$  is the generator of the last summand  $\mathbb{N}$ ), the chart is  $m_a, m_b, m_\pi \mapsto t_a, t_b, \pi_{K'}$ . Consider an embedding  $M_{A,B} \hookrightarrow M_{A,B}^w := e^{-1}\mathbb{N}[A] \oplus \mathbb{N}[B]$ ,  $m_a, m_b, m_\pi \mapsto m_a, m_b, \Pi m_a^{1/e}$ . Its image is formed by those  $\Pi m_a^{n_a/e} \Pi m_b^{n_b}$ ,  $n_a, n_b \in \mathbb{N}$ , such that  $n_a - n_{a'} \in e\mathbb{Z}$  for any  $a, a' \in A$ . Thus  $M_{A,B}$  is saturated. Now the log scheme  $(\bar{U}_{O_{K'}}, \mathcal{M})$  is evidently log regular in the sense of [K2] 2.1, hence  $\mathcal{M} \xrightarrow{\sim} \mathcal{O}_{\bar{U}_{O_{K'}}} \cap j_* \mathcal{O}_{U_{K'}}^\times$  by [K2] 11.6, q.e.d.<sup>16</sup>

The reference to [K2] can be replaced by the next explicit argument: It suffices to show that the map of sheaves  $\mathcal{M}/\mathcal{O}_{\bar{U}_{O_{K'}}}^\times \rightarrow (\mathcal{O}_{\bar{U}_{O_{K'}}} \cap j_* \mathcal{O}_{U_{K'}}^\times)/\mathcal{O}_{\bar{U}_{O_{K'}}}^\times$  is an isomorphism. The r.h.s. is the sheaf  $\mathcal{D}$  of effective Cartier divisors supported on  $\bar{U}_{O_{K'}} \setminus U_{K'}$ . Let  $\mathcal{D}^w \supset \mathcal{D}$  be the sheaf of the corresponding effective Weil divisors. For  $x \in \bar{U}_{O_{K'}}$ , the fiber  $(\mathcal{M}/\mathcal{O}_{\bar{U}_{O_{K'}}}^\times)_x$  is the quotient  $M_{A_x, B_x}$  of  $M_{A,B}$ , where  $A_x \subset A$ ,  $B_x \subset B$  consist of those  $a, b$  such that  $t_a, t_b$  vanish at  $x$ . The map  $M_{A_x, B_x} \rightarrow \mathcal{D}_x$  extends to an isomorphism  $M_{A_x, B_x}^w \xrightarrow{\sim} \mathcal{D}_x^w$ , which identifies a generator  $m_a^{1/e}$  with the *reduced* divisor  $D_a$  of  $t_a$ ,  $m_b$  with  $D_b := \text{div}(t_b)$ . Thus  $M_{A_x, B_x} \hookrightarrow \mathcal{D}_x$ . To show that  $\hookrightarrow$  is an isomorphism, we need to check that if  $D = \sum n_a D_a + \sum n_b D_b$  is a Cartier divisor at  $x$ , then  $n_a - n_{a'} \in e\mathbb{Z}$  for any  $a, a' \in A_x$ . We can assume that  $A = \{a, a'\}$ ,  $B = C = \emptyset$ , so  $\bar{U}_{O_{K'}}$  is a semi-stable curve over  $O_{K'}$ . The exceptional divisor of its minimal desingularization  $\tilde{U}_{O_{K'}}$  is a chain of  $e-1$  projective lines  $P_1, \dots, P_{e-1}$  with self-intersection indices  $(P_i, P_i) = -2$ . Let  $\tilde{D} = n_a D_a^b + n_1 P_1 + \dots + n_{e-1} P_{e-1} + n_{a'} D_{a'}^b$  be the pullback of  $D$  to  $\tilde{U}_{O_{K'}}$ ; here  $D_a^b, D_{a'}^b$  are strict transforms of  $D_a, D_{a'}$ . One has  $(\tilde{D}, P_i) = 0$ , i.e.,  $n_{i-1} - 2n_i + n_{i+1} = 0$  or  $n_i - n_{i-1} = n_{i+1} - n_i$ , where  $n_0 := n_a, n_e := n_{a'}$ . Thus  $n_{a'} - n_a = e(n_1 - n_a) \in e\mathbb{Z}$ , and we are done.  $\square$

<sup>15</sup>We replace one  $t_a \in \mathcal{O}(\bar{U}_{O_{K'}})$  by  $t_a \pi_{K'}^e / \pi_{K_U}$ .

<sup>16</sup>I am grateful to Luc Illusie for the proof.

4.2. Set  $\Omega_{\langle V, \bar{V} \rangle}^a := \Lambda_{\mathcal{O}_{\bar{V}}}^a \Omega_{\langle V, \bar{V} \rangle}$ . Consider (4.1.1) as a 2-step filtration on  $\Omega_{\langle V, \bar{V} \rangle}$ ; it splits locally since  $\Omega_{\langle V, \bar{V} \rangle}$  is locally free. Passing to derived exterior powers, we get for any  $m$  a finite increasing filtration  $I$  on  $\mathrm{gr}_F^m L\Omega_{\langle \hat{V}, \bar{V} \rangle} = (L\Lambda_{\mathcal{O}_{\bar{V}}}^m \Omega_{\langle V, \bar{V} \rangle})[-m]$  with  $\mathrm{gr}_a^I \mathrm{gr}_F^m L\Omega_{\langle \hat{V}, \bar{V} \rangle} = \Omega_{\langle V, \bar{V} \rangle}^a \otimes_{\mathcal{O}_{\bar{K}}} \mathrm{gr}_F^{m-a} \mathbf{A}_{\mathrm{dR}}[-a]$ , hence on  $\mathrm{gr}_F^m R\Gamma_{\mathrm{dR}}^{\natural}(V, \bar{V})$  with

$$(4.2.1) \quad \mathrm{gr}_a^I \mathrm{gr}_F^m R\Gamma_{\mathrm{dR}}^{\natural}(V, \bar{V}) = R\Gamma(\bar{V}, \Omega_{\langle V, \bar{V} \rangle}^a) \otimes_{\mathcal{O}_{\bar{K}}}^L \mathrm{gr}_F^{m-a} \mathbf{A}_{\mathrm{dR}}[-a].$$

Let  $\mathcal{G}^a$  be the h-sheafification (see (2.6.1)) of the complex of presheaves  $(V, \bar{V}) \mapsto R\Gamma(\bar{V}, \Omega_{\langle V, \bar{V} \rangle}^a)$  on  $\mathcal{V}ar_{\bar{K}}^{\mathrm{ss}}$ . This is a complex of h-sheaves of  $\mathcal{O}_{\bar{K}}$ -modules on  $\mathcal{V}ar_{\bar{K}}$ . Its cohomology  $H^b \mathcal{G}^a$  is h-sheafification of the presheaf  $(V, \bar{V}) \mapsto H^b(\bar{V}, \Omega_{\langle V, \bar{V} \rangle}^a)$  on  $\mathcal{V}ar_{\bar{K}}^{\mathrm{ss}}$ . Our  $I$  is a filtration on the presheaf  $(V, \bar{V}) \mapsto \mathrm{gr}_F^m R\Gamma_{\mathrm{dR}}^{\natural}(V, \bar{V})$ ; passing to h-sheafification, we get a finite filtration  $I$  on  $\mathrm{gr}_F^m \mathcal{A}_{\mathrm{dR}}^{\natural}$  with span  $[0, m]$  and

$$(4.2.2) \quad \mathrm{gr}_a^I \mathrm{gr}_F^m \mathcal{A}_{\mathrm{dR}}^{\natural} = \mathcal{G}^a \otimes_{\mathcal{O}_{\bar{K}}}^L \mathrm{gr}_F^{m-a} \mathbf{A}_{\mathrm{dR}}[-a].$$

Notice that the bottom cohomology  $H^0$  of the bottom term  $I_0 = \mathrm{gr}_0^I$  is the constant sheaf  $\mathcal{O}_{\bar{K}}$  and  $\mathrm{Cone}(\mathrm{gr}_F^m \mathbf{A}_{\mathrm{dR}} \rightarrow \mathrm{gr}_F^m \mathcal{A}_{\mathrm{dR}}^{\natural}) = \mathrm{gr}_F^m \mathcal{A}_{\mathrm{dR}}^{\natural} / H^0 I_0$ . Therefore, by (4.2.2), the Poincaré lemma follows from the next assertion:

**Theorem.** *The cohomology  $H^b \mathcal{G}^a$  are h-sheaves of  $\mathbb{Q}$ - (hence  $\bar{K}$ -) vector spaces for  $(a, b) \neq (0, 0)$ .*

*Remark.* The p-divisibility of  $H^b \mathcal{G}^0$ ,  $b \neq 0$ , was first proved by Bhatt [Bh1] 8.0.1.

*Exercise.* Consider a presheaf  $(V, \bar{V}) \mapsto R\Gamma(\bar{V}, L\Omega_{\langle \hat{V}, \bar{V} \rangle} / (\bar{K}, \mathcal{O}_{\bar{K}}))$ ; let  $\mathcal{A}_{\mathrm{dR}}^{\mathrm{naive}}$  be its h-sheafification. One has an evident map  $\mathrm{Cone}(F^1 \mathbf{A}_{\mathrm{dR}} \rightarrow \mathcal{A}_{\mathrm{dR}}^{\natural}) \rightarrow \mathcal{A}_{\mathrm{dR}}^{\mathrm{naive}}$ . Show that the theorem implies that it is a filtered quasi-isomorphism, i.e., the triangle  $F^1 \mathbf{A}_{\mathrm{dR}} \rightarrow \mathcal{A}_{\mathrm{dR}}^{\natural} \rightarrow \mathcal{A}_{\mathrm{dR}}^{\mathrm{naive}}$  is exact in the filtered derived category of h-sheaves.

4.3. We deduce the above theorem from a more concrete assertion. As in 4.1, for an ss-pair  $(U, \bar{U})$  over  $K$  we have the locally free  $\mathcal{O}_{\bar{U}}$ -module of log differentials  $\Omega_{\langle U, \bar{U} \rangle} := \Omega_{(U, \bar{U}) / (K_U, \mathcal{O}_{K_U})}$  and its exterior powers  $\Omega_{\langle U, \bar{U} \rangle}^a := \Lambda^a \Omega_{\langle U, \bar{U} \rangle}$ .

Let  $f : (U', \bar{U}') \rightarrow (U, \bar{U})$  be a map in  $\mathcal{V}ar_K^{\mathrm{ss}}$  or  $\mathcal{V}ar_{\bar{K}}^{\mathrm{ss}}$ . We say that  $f$  is (Hodge) *p-negligible* if the morphisms  $(\tau_{>0} R\Gamma(\bar{U}, \mathcal{O}_{\bar{U}})) \otimes^L \mathbb{Z}/p \rightarrow (\tau_{>0} R\Gamma(\bar{U}', \mathcal{O}_{\bar{U}'})) \otimes^L \mathbb{Z}/p$  and  $R\Gamma(\bar{U}, \Omega_{\langle U, \bar{U} \rangle}^a) \otimes^L \mathbb{Z}/p \rightarrow R\Gamma(\bar{U}', \Omega_{\langle U', \bar{U}' \rangle}^a) \otimes^L \mathbb{Z}/p$ ,  $a > 0$ , in  $D^b(\mathcal{O}_{K_U}/p)$ , resp.  $D^b(\mathcal{O}_{\bar{K}}/p)$ , vanish.

*Remark.* For  $(U, \bar{U}) \in \mathcal{V}ar_K^{\mathrm{ss}}$  and a point  $K_U \rightarrow \bar{K}$ , one has  $R\Gamma(\bar{U}_{\mathcal{O}_{\bar{K}}}, \Omega_{\langle U_{\bar{K}}, \bar{U}_{\mathcal{O}_{\bar{K}}} \rangle}^a) = R\Gamma(\bar{U}, \Omega_{\langle U, \bar{U} \rangle}^a) \otimes_{\mathcal{O}_K}^L \mathcal{O}_{\bar{K}}$ . Therefore the base change functor  $\mathcal{V}ar_K^{\mathrm{ss}} \rightarrow \mathcal{V}ar_{\bar{K}}^{\mathrm{ss}}$  (see (2.2.2)) preserves *p-negligible* maps.

**Theorem.** *Every  $U \in \mathcal{V}ar_{\bar{K}}^{\mathrm{ss}}$  admits a *p-negligible* h-covering. Ditto for  $\bar{K}$ -pairs.*

The theorem implies the one in 4.2: Indeed, the  $\bar{K}$ -assertion shows that one has  $(\tau_{>0} \mathcal{G}^0) \otimes^L \mathbb{Z}/p = 0$  and  $\mathcal{G}^a \otimes^L \mathbb{Z}/p = 0$  for  $a > 0$ ; since for a complex  $\mathcal{G}$  the multiplication by  $p$  on  $H^* \mathcal{G}$  is invertible if and only if  $\mathcal{G} \otimes^L \mathbb{Z}/p = 0$ , we are done. Thus it yields the Poincaré lemma.

The above remark shows that the  $K$ -version of the theorem implies the  $\bar{K}$ -one. The proof of the  $K$ -version takes the rest of the section.

4.4. For the rest of §4, “pair” means “arithmetic  $K$ -pair” (see 2.2). We need further input from de Jong. A morphism  $f : (C, \bar{C}) \rightarrow (S, \bar{S})$  of pairs is said to be a *family of pointed curves* (over  $(S, \bar{S})$ ) if the map  $\bar{C}_S := f^{-1}(S) \rightarrow S$  is smooth of relative dimension 1 with irreducible geometric fibers, and  $D_{fS} := \bar{C}_S \setminus C$ , viewed as a reduced scheme, is étale over  $S$ . Such an  $f$  is *semi-stable* if, in addition,  $\bar{C}/\bar{S}$  is a semi-stable family of curves, and the closure  $D_f$  of  $D_{fS}$  in  $\bar{C}$  (the *horizontal divisor*), viewed as a reduced scheme, is étale over  $\bar{S}$  and intersects each fiber of  $f$  at smooth points. A section  $e : (S, \bar{S}) \rightarrow (C, \bar{C})$  of  $f$  is said to be *nice* if  $e(\bar{S})$  intersects fibers of  $f$  at smooth points and  $D_f \cap e(\bar{S}) = \emptyset$ . Families of pointed curves over  $(S, \bar{S})$  form a category  $\mathcal{C}_{(S, \bar{S})}$  in the obvious manner, and a morphism of bases  $\psi : (S', \bar{S}') \rightarrow (S, \bar{S})$  yields an evident pullback functor  $\mathcal{C}_{(S, \bar{S})} \rightarrow \mathcal{C}_{(S', \bar{S}')}$  which preserves semi-stable families. A morphism  $f' \rightarrow f$  in  $\mathcal{C}_{(S, \bar{S})}$  is called *alteration* if  $(C', \bar{C}')$  is an alteration of  $(C, \bar{C})$ ; it is a *semi-stable alteration* (of  $f$ ) if, in addition,  $f'$  is semi-stable.

**Theorem.** (a) Any family  $f : (C, \bar{C}) \rightarrow (S, \bar{S})$  of pointed curves with  $f : \bar{C} \rightarrow \bar{S}$  projective admits a semi-stable alteration  $f'$   $h$ -locally over  $(S, \bar{S})$ .

(b) One can find  $f'$  as above which has a nice section  $e$ . Moreover, for a given closed subscheme  $P \subset \bar{C}$  such that  $f(P) = \bar{S}$  and  $P \cap \bar{C}_S \subset C$ , one can find  $e$  such that the map  $\bar{C}' \rightarrow \bar{C}$  sends  $e(\bar{S})$  to  $P$ .

(c) For any semi-stable family of pointed curves  $f : (C, \bar{C}) \rightarrow (S, \bar{S})$  with  $(S, \bar{S})$  a strict ss-pair, there exists a semi-stable alteration  $m : (C, \tilde{C}) \rightarrow (C, \bar{C})$  of  $f$  with  $m|_C = \text{id}_C$  such that  $m : \tilde{C} \rightarrow \bar{C}$  is an isomorphism over smooth points of  $f$  and  $(C, \tilde{C})$  is an ss-pair.

*Proof.* (c) is [dJ1] 3.6. (a) follows from [dJ2] 2.4 (i),(ii) except that de Jong does not care to control the domain of smoothness of the semi-stable alteration of  $f$ . A miniscule modification of his argument permits to do this; see Appendix 1. Alternatively, (a) follows directly from a far more precise result of Temkin [T] 1.5.<sup>17</sup>

Let us check (b). Every pair has a canonical alteration by the union of normalizations of its irreducible components, so we assume all the way that  $\bar{S}$  is normal and irreducible. Since  $P$  as in (b) exists  $h$ -locally on  $(S, \bar{S})$ ,<sup>18</sup> we can assume it is given. Replacing  $(S, \bar{S})$  by its alteration  $(P_S, P)$ , we get a section  $e$  of  $f$  with image in  $P$ . Set  $C^b := C \setminus e(S)$ . Then  $(C^b, \bar{C}) \rightarrow (S, \bar{S})$  is a family of pointed curves; let  $f^b : (C^b, \bar{C}') \rightarrow (S, \bar{S})$  be its semi-stable alteration as in (a). Let  $D_e$  be the closure in  $\bar{C}'$  of the preimage  $D_{eS}$  of  $e(S)$ . Then  $D_e$  is an étale covering of  $\bar{S}$ .<sup>19</sup> Let  $C'$  be the preimage of  $C \subset \bar{C}$  in  $\bar{C}'$ ; then  $(C', \bar{C}') \rightarrow (S, \bar{S})$  is a semi-stable alteration of  $(C, \bar{C}) \rightarrow (S, \bar{S})$ . Replacing  $(S, \bar{S})$  by its alteration  $(D_{eS}, D_e)$ , we get a nice section of  $(C', \bar{C}')$  which sits over  $e$ , hence over  $P$ .  $\square$

*Remark.* In (c), every nice section of  $(C, \bar{C})$  lifts to a nice section of  $(C, \tilde{C})$ .

**Corollary.** Any pair  $(U, \bar{U})$  has an  $h$ -covering by ss-pairs  $(C, \bar{C})$ ,  $\dim C = \dim U$ , for which there is a semi-stable family of pointed curves  $f : (C, \bar{C}) \rightarrow (S, \bar{S})$  with a nice section such that  $(S, \bar{S})$  is an ss-pair and  $C$  is affine over  $S$  (i.e.,  $f(D_f) = \bar{S}$ ).

<sup>17</sup>To bring our datum to Temkin’s setting, one flattens  $\bar{C}/\bar{S}$  and  $D_f/\bar{S}$  using [RG] 5.2.2 and replaces  $\bar{S}$  by its normalization.

<sup>18</sup>Indeed, one can cover  $S$  by open subsets  $\{S_\alpha\}$  such that  $P$  exists for  $(C_{S_\alpha}, \bar{C}) \rightarrow (S_\alpha, \bar{S})$ .

<sup>19</sup>Since  $D_{f^b}$  is étale over  $\bar{S}$ ,  $D_{eS}$  is open and closed in  $D_{f^b S}$ ,  $f^b(D_{eS}) = S$ , and  $\bar{S}$  is normal.

*Proof.* It suffices to find an h-covering of  $(U, \bar{U})$  by pairs  $(C, \bar{C})$  with  $\dim C = \dim U$  for which there exists a family of pointed curves  $f : (C, \bar{C}) \rightarrow (S, \bar{S})$  with  $C$  affine over  $S$  and projective  $\bar{S}$ ,  $\bar{C}$ . The theorem transforms it then, with an input from Remark (i) in 2.5 to preserve the dimension and de Jong's theorem in 2.3 to alter  $(S, \bar{S})$  from (b) into a strict ss-pair, into a datum with all promised properties.

By de Jong's theorem in 2.3, we can assume that  $(U, \bar{U})$  is an ss-pair and  $\bar{U}$  is projective and irreducible;<sup>20</sup> set  $d = \dim U$ . Pick any closed point  $u \in U$ . It suffices to find an open neighborhood  $U' \subset U$  of  $U$ , an alteration  $(C, \bar{C})$  of  $(U', \bar{U})$ , and a family of curves  $f : (C, \bar{C}) \rightarrow (S, \bar{S})$  such that  $f(D_f) = \bar{S}$ .

Embed  $\bar{U}$  into a projective space  $\mathbb{P}_{O_K}^N$ . By Bertini, there is a plane  $H \subset \mathbb{P}^N$  defined over  $K$  of codimension  $d$  such that  $u \notin H$ ,  $H$  intersects  $\bar{U}_K$  transversally,  $H \cap \bar{U}_K \subset U$ , and the codimension  $d-1$  plane which contains  $H$  and  $u$ , is transversal to  $\bar{U}_K$  and  $\bar{U}_K \setminus U$ . Let  $m : \bar{C} \rightarrow \bar{U}$  be the blowup at  $\bar{U} \cap H_{O_K}$ ,  $p : \bar{C} \rightarrow \mathbb{P}_{O_K}^{d-1}$  be the projection defined by  $H$ , and  $\bar{C} \xrightarrow{f} \bar{S} \rightarrow \mathbb{P}_{O_K}^{d-1}$  be the Stein factorization of  $p$  (so  $\bar{S} = \text{Spec } p_* \mathcal{O}_{\bar{C}}$ ),  $D \subset \bar{C}$  be the union of  $m^{-1}(\bar{U} \setminus U)$  and the exceptional divisor (viewed as a reduced scheme),  $S \subset \bar{S}_K$  be the maximal open subset over which  $f$  is smooth and  $f|_D$  is étale. Set  $C := f^{-1}(S) \setminus D$  and  $U' := m(C)$ ; notice that  $m|_C : C \xrightarrow{\sim} U'$ . Then  $U'$ ,  $(C, \bar{C})$ ,  $f$  satisfy the promised properties (one has  $f(D_f) = \bar{S}$  since  $D_f$  contains the exceptional divisor), q.e.d.  $\square$

4.5. Let us return to the proof of the theorem in 4.3. We use induction by  $\dim U$ . By the corollary in 4.4, we can replace  $(U, \bar{U})$  by  $(C, \bar{C})$  as in loc. cit., so we have  $f : (C, \bar{C}) \rightarrow (S, \bar{S})$  with a nice section  $e$  and  $C$  affine over  $S$ . Notice that  $(C, \bar{C})$  is log smooth over  $(S, \bar{S})$  and the line bundle  $\omega_f := \Omega_{(C, \bar{C})/(S, \bar{S})}$  equals  $f^!(\mathcal{O}_{\bar{S}}[-1] \otimes \mathcal{O}_{\bar{C}}(D_f))$ .

**Key lemma.** *h-locally over  $(S, \bar{S})$ , one can find a semi-stable alteration  $\phi : f' \rightarrow f$  together with a nice section  $e'$  that lifts  $e$  such that  $(C', \bar{C}')$  is an ss-pair and the pullback maps  $\phi^* : R^1 f_* \mathcal{O}_{\bar{C}} \rightarrow R^1 f'_* \mathcal{O}_{\bar{C}'}$ ,  $f_* \omega_f \rightarrow f'_* \omega_{f'}$  are divisible by  $p$ .*<sup>21</sup>

For a proof, see 4.6. Assuming it for the moment, let us finish the proof of the theorem in 4.3. By Remark (i) in 2.5, we can assume that the h-localization of  $(S, \bar{S})$  in Key Lemma does not change  $\dim S$ . We will show that for some h-covering  $(S', \bar{S}')$  of  $(S, \bar{S})$  the composition  $(C', \bar{C}')_{(S', \bar{S}')} \rightarrow (C', \bar{C}') \xrightarrow{\phi} (C, \bar{C})$  is  $p$ -negligible.

For any  $a$  consider the exact sequence

$$(4.5.1) \quad 0 \rightarrow f^* \Omega_{(S, \bar{S})}^a \rightarrow \Omega_{(C, \bar{C})}^a \rightarrow (f^* \Omega_{(S, \bar{S})}^{a-1}) \otimes \omega_f \rightarrow 0.$$

The section  $e$  splits off  $\Omega_{(S, \bar{S})}^a \hookrightarrow Rf_* \Omega_{(C, \bar{C})}^a$  as a direct summand whose complement is  $\text{Cone}(\partial_C)$ , where  $\partial_C : \Omega_{(S, \bar{S})}^{a-1} \otimes f_* \omega_f \rightarrow \Omega_{(S, \bar{S})}^a \otimes R^1 f_* \mathcal{O}_{\bar{C}}$ , is the boundary map for (4.5.1) (one has  $R^1 f_* \omega_f = 0$  since  $f(D_f) = \bar{S}$ ). There is a similar splitting in case of  $f'$  provided by  $e'$ , and the map  $\phi^* : Rf_* \Omega_{(C, \bar{C})}^a \rightarrow Rf'_* \Omega_{(C', \bar{C}')}^a$  is compatible with the direct sum decompositions. Now  $\phi^*$  is divisible by  $p$  on the second summand: Indeed, Key Lemma asserts that the morphism of two-term complexes  $\phi^* : \text{Cone}(\partial_C) \rightarrow \text{Cone}(\partial_{C'})$  is divisible by  $p$  on each term; since these are morphisms of vector bundles on  $O_K$ -flat  $\bar{S}$ , our  $p^{-1}\phi^*$  is uniquely defined and commutes

<sup>20</sup>We only need that  $\bar{U}$  is projective and normal, and that  $U$  is smooth.

<sup>21</sup>As elements of the groups  $\text{Hom}_{\mathcal{O}_{\bar{S}}} (R^1 f_* \mathcal{O}_{\bar{C}}, R^1 f'_* \mathcal{O}_{\bar{C}'})$ ,  $\text{Hom}_{\mathcal{O}_{\bar{S}}} (f_* \omega_f, f'_* \omega_{f'})$ .

with the differentials. Thus the map  $\phi^* \otimes \text{id}_{C_1} : \text{Cone}(\partial_C) \otimes C_1 \rightarrow \text{Cone}(\partial_{C'}) \otimes C_1$ , where  $C_1 := \text{Cone}(p : \mathbb{Z} \rightarrow \mathbb{Z})$ , is homotopic to zero. Apply  $R\Gamma(\bar{S}, \cdot)$  and use the induction assumption to treat the first summand  $R\Gamma(\bar{S}, \Omega_{\bar{S}}^a)$ ; we are done.  $\square$

**4.6. Proof of Key Lemma.** Consider the relative Picard  $\bar{S}$ -schemes  $J := \text{Pic}^0(\bar{C}/\bar{S})$  and  $J^b := \text{Pic}^0((\bar{C}, D_f)/\bar{S})$ : the first scheme parametrizes line bundles  $\mathcal{L}$  on  $\bar{C}$  such that the restriction of  $\mathcal{L}$  to the normalization of each irreducible component of any geometric fiber of  $f$  has degree 0; the second one parametrizes pairs  $(\mathcal{L}, \gamma)$ , where  $\mathcal{L}$  is as above and  $\gamma$  is a trivialization of  $\mathcal{L}|_{D_f}$ . Since  $(\bar{C}, D_f)$  is a semi-stable  $\bar{S}$ -family of  $d$ -pointed curves,  $d := \deg(D_f)$ ,  $J$  and  $J^b$  are semi-abelian schemes (see [R]);  $J^b$  is an extension of  $J$  by a torus  $\mathbb{G}_m^{D_f}/\mathbb{G}_m$ .

Over  $S$  our  $J^b$  is a generalized Jacobian; let  $i : C \rightarrow J_S^b$  be the Abel-Jacobi map  $i : C \rightarrow J_S^b$ ,  $x \mapsto \mathcal{O}_{\bar{C}}(x - e)$ . Let  $C^\sim \rightarrow C$  be the  $i$ -pullback of the multiplication by  $p$  isogeny  $p_{J^b} : J^b \rightarrow J^b$ , and  $\bar{C}^\sim \rightarrow \bar{C}$  be the normalization of  $\bar{C}$  in  $C^\sim$ . Then  $f^\sim : (C^\sim, \bar{C}^\sim) \rightarrow (S, \bar{S})$  is a family of pointed curves, which is an alteration of  $f$ . By the theorem in 4.4, h-locally over  $(S, \bar{S})$  there is a semi-stable alteration  $f'$  of  $f^\sim$  with  $(C', \bar{C}')$  semi-stable and equipped with a nice section  $e'$  which lies over  $e$ . Let us check that the alteration  $\phi : f' \rightarrow f$  satisfies the conditions of Key Lemma.

Set  $J' := \text{Pic}^0(\bar{C}'/\bar{S})$  and  $J'^b := \text{Pic}^0((\bar{C}', D_{f'})/\bar{S})$ . We have the pullback morphisms  $\phi^* : J \rightarrow J'$ ,  $J^b \rightarrow J'^b$  of our semi-abelian schemes; over  $S$  we have the norm maps  $\phi_{*S} : J'_S \rightarrow J_S$ ,  $J'^b_S \rightarrow J^b_S$ . Both are compatible with the projections  $J^b \rightarrow J$ ,  $J'^b \rightarrow J'$ .

Since  $\bar{S}$  is normal, for any semi-abelian  $\bar{S}$ -schemes  $A, B$  one has (see [FC] I 2.7)

$$(4.6.1) \quad \text{Hom}(A, B) \xrightarrow{\sim} \text{Hom}(A_S, B_S).$$

Thus  $\phi_{*S}$  extends to morphisms  $\phi_* : J' \rightarrow J$ ,  $J'^b \rightarrow J^b$ .

Notice that  $R^1 f_* \mathcal{O}_{\bar{C}}$  is the Lie algebra of  $J$ , and, by Serre duality,  $f_* \omega_f$  is dual to the Lie algebra of  $J^b$ ; the same is true for  $f'$ . Our  $\phi^* : R^1 f_* \mathcal{O}_{\bar{C}} \rightarrow R^1 f'_* \mathcal{O}_{\bar{C}'}$  is the Lie algebra map for  $\phi^* : J \rightarrow J'$ , and  $\phi^* : f_* \omega_f \rightarrow f'_* \omega_{f'}$  is the map between the duals to the Lie algebras for  $\phi_* : J^b \rightarrow J'^b$  (this is true over  $S$ , hence everywhere since  $\bar{S}$  is  $O_K$ -flat).

By construction,  $\phi_{*S} : J'^b_S \rightarrow J^b_S$  factors through  $p_{J^b}$  over  $S$ , i.e., it is divisible by  $p$  in  $\text{Hom}(J'^b_S, J^b_S)$ . By (4.6.1),  $\phi_*$  is divisible by  $p$  in  $\text{Hom}(J'^b, J^b)$ . Passing to Lie algebras, we see that  $\phi^* : f_* \omega_f \rightarrow f'_* \omega_{f'}$  is divisible by  $p$ . Similarly,  $\phi_{*S} : J'_S \rightarrow J_S$  is divisible by  $p$ . Notice that  $J_S, J'_S$ , being Jacobians of smooth projective curves, are self-dual abelian schemes, and  $\phi_S^* : J_S \rightarrow J'_S$  is dual to  $\phi_{*S}$ . Hence  $\phi_S^*$  is divisible by  $p$ . So, by (4.6.1),  $\phi^* : J \rightarrow J'$  is divisible by  $p$ . Passing to Lie algebras, we see that  $\phi^* : R^1 f_* \mathcal{O}_{\bar{C}} \rightarrow R^1 f'_* \mathcal{O}_{\bar{C}'}$  is divisible by  $p$ , q.e.d.  $\square$

#### APPENDIX.

Below is a proof of part (a) in the theorem from 4.4. It follows closely de Jong's argument from §§2–3 of [dJ2] with a minor change of the lemma below; we refer the reader to sections of [dJ2] for details.

(i) ([dJ2] 2.10) One can assume that  $\bar{S}$  is irreducible. By [RG] 5.2.2, there is a canonical modification of  $\bar{C}$ , which is projective and is an isomorphism over  $S$ , such that the strict transforms of  $\bar{C}$  and  $D_f$  are flat over  $\bar{S}$ . Passing to them, we can assume that *all fibers of  $f$  have dimension 1, of  $f|_{D_f}$  have dimension 0*.

(ii) ([dJ2] 3.4–3.5) We say that a family of pointed curves is *good* if irreducible components of all its geometric fibers are curves whose normalization has genus  $\geq 2$ . A good alteration is an alteration with good source.

**Lemma.**  *$f$  admits a good alteration  $h$ -locally over  $(S, \bar{S})$ .*

*Proof of Lemma.* It suffices to find for any closed point  $s$  in  $S$  its open neighborhood  $S_{(s)} \subset S$  and an alteration  $(S'_{(s)}, \bar{S}')$  of  $(S_{(s)}, \bar{S})$  such that the pullback of  $f$  to  $(S'_{(s)}, \bar{S}')$  admits a good alteration. To do this, we define by induction a strictly increasing sequence of open subsets  $\emptyset = V_0 \subset V_1 \subset \dots$  of  $\bar{S}$  and a sequence of finite extensions  $F = F_0 \subset F_1 \subset \dots$  of the field  $F$  of rational functions on  $\bar{C}$  such that the normalization  $\bar{C}_i$  of  $\bar{C}$  in  $F_i$  has next properties: (a) the map  $\bar{C}_i \rightarrow \bar{S}$  is smooth at  $s$ , (b) the map  $\pi_i : \bar{C}_i \rightarrow \bar{C}$  is étale at  $D_{f,s}$ , (c) the normalizations of irreducible components of geometric fibers of  $\bar{C}_i$  over points of  $V_i$  have genus  $\geq 2$ . There is an open neighborhood  $U_i \subset S$  of  $s$  over which  $\bar{C}_i$  is smooth and  $\pi_i$  is étale at  $D_f$ . The induction stops when  $V_n = \bar{S}$ ; set  $S_{(s)} = U_n$ . The promised good alteration is  $(\pi_n^{-1}(C), \bar{C}_n)$  fibered over the normalization  $(S'_{(s)}, \bar{S}')$  of  $(S_{(s)}, \bar{S})$  in  $F_n$ .

Let  $x$  be the closed point of the closure of  $s$  in  $\bar{S}$ . The induction produces simultaneously an auxiliary sequence of finite subsets  $T_0 \subset T_1 \subset \dots$  of closed points of  $\bar{C}_x$ ; it starts with  $T_0 :=$  the union of  $D_{f,x}$  and the set of nonregular points of  $\bar{C}_x$ . The induction step: suppose we have  $V_{i-1}, F_{i-1}, T_{i-1}$ ; let us construct  $V_i, F_i, T_i$  assuming that  $V_{i-1} \neq \bar{S}$ . Let  $y$  be any closed point in  $\bar{S} \setminus V_{i-1}$ . Since  $\bar{S}$  is projective, there is an affine open  $V$  which contains  $x$  and  $y$ . Let  $\mathcal{L}$  be a very ample line bundle on  $\bar{C}$ . Replacing it by a sufficiently high power, we can assume that  $\Gamma(\bar{C}_V, \mathcal{L}) \twoheadrightarrow \Gamma(\bar{C}_x, \mathcal{L}) \times \Gamma(\bar{C}_y, \mathcal{L})$ . One can find a finite unramified extension<sup>22</sup>  $K'$  of  $K$  with residue field  $k'$  and two sections  $\gamma_1, \gamma_2 \in \Gamma(\bar{C}_V, \mathcal{L}) \otimes_{O_K} O_{K'}$  which do not vanish at the generic points of irreducible components of  $\bar{C}_x, \bar{C}_y$ , such that  $t = \gamma_1/\gamma_2$  yields generically étale finite maps  $t_x : \bar{C}_x \otimes k' \rightarrow \mathbb{P}_x^1 \otimes k', t_y : \bar{C}_y \otimes k' \rightarrow \mathbb{P}_y^1 \otimes k'$  étale over  $\{0, 1, \infty\}$  and such that  $t_x(T_{i-1}) \cap \{0, 1, \infty\} = \emptyset$ . Pick  $\ell \geq 5$  prime to  $p$ , and let  $F_i$  be an extension of  $F_{i-1}$  generated by  $K', \mu_\ell, t^{1/\ell}$ , and  $(1-t)^{1/\ell}$ . Let  $T_i$  be the union of  $T_{i-1}$  and the set of ramification points of  $t_x$ . The normalization  $\bar{C}_i$  of  $\bar{C}$  in  $F_i$  satisfies (a), (b), and satisfies (c) over some open set  $V_i$  which contains  $V_{i-1}$  and  $y$ . We are done.  $\square$

(iii) It remains to show that *every good  $f$  admits a semi-stable alteration after a possible alteration of the base*. The genus of the generic fiber of  $f$  is  $\geq 2$ , so  $(\bar{C}_S, D_{f,S})$  is a stable  $n$ -pointed curve over  $S$  (where  $n$  is the degree of  $D_{f,S}$  over  $S$ ). The Deligne-Mumford stack of stable  $n$ -pointed curves is proper, so, after replacing  $(S, \bar{S})$  by an alteration, we can assume that  $(\bar{C}_S, D_{f,S})$  extends to a stable  $n$ -pointed curve  $(\bar{C}', D_{f'})$  over  $\bar{S}$  (see [dJ2] 3.8). We have a semi-stable family of pointed curves  $f' : (C', \bar{C}') \rightarrow (S, \bar{S})$ ,  $C' := \bar{C}' \setminus D_{f'}$ . By [dJ2] 3.10, the goodness of  $f$  implies that, after a possible alteration of  $\bar{S}$ , the evident morphism  $\bar{C}'_S \rightarrow \bar{C}_S$  extends to a morphism  $f' \rightarrow f$ , and we are done.  $\square$

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<sup>22</sup>If the residue field of  $K$  is infinite, one can take  $K' = K$ .

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